Submanifolds with curvature normals of constant length and the Gauss map

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Abstract

Let $f : M^n \to \mathbb{R}^N$ be a submanifold of euclidean space and let $\nu_0(M)$ be the maximal parallel and flat subbundle of the normal bundle $\nu(M)$. Then $rank_f(M)$ is the dimension over M of $\nu_0(M)$. One can define the curvature normals of M with respect to $\nu_0(M)$. Assume that M is simply connected and complete (with the induced metric) and that $n \geq 2$. Assume furthermore that the curvature normals of M have constant length and that M is full and irreducible as a submanifold. Then M has constant principal curvatures (under the additional mild assumption that either the number of principal curvatures is constant on M or that the local and global rank coincide). This result is not true for curves or without assuming the completeness of M. In particular, we prove that a full irreducible and complete submanifold of euclidean space with flat normal bundle (different from a curve) is isoparametric if and only if its curvature normals have constant length (or, equivalently, the distance to any focal hyperplane is constant, or the third fundamental form has constant eigenvalues).

1 Introduction

The theory of isoparametric submanifolds of euclidean space is a source of important examples and insights. It has many links with other branches of differential geometry and topology (e.g. symmetric spaces, polar actions, Coxeter groups, sphere bundles, Tits buildings, theory of minimal submanifolds, etc) [C, E, K, HL, PT, T1, T2]. For the study of isoparametric submanifolds one needs to consider global data. Though, a posteriori, a local isoparametric submanifold turns out to extend to a complete one. An isoparametric submanifold M of euclidean space has the property that the distance to any of its focal hyperplanes is constant on M or, equivalently, the length of any curvature normal is constant. This property can be read off from the (possibly degenerate) metric $\langle B\,,\,\rangle$ induced by the Gauss map into the Grassmannian. Namely, the lengths of the curvature normals are given by the eigenvalues of the so called third fundamental form B (see §2). The first natural question which arises is whether a submanifold of euclidean space with flat normal bundle and curvature normals of constant length must be isoparametric (or, equivalently, whether the curvature normals are parallel in the normal connection). If M has not flat normal bu! ndle, the curvature normals can be defined for $\nu_0(M)$, the maximal parallel and flat subbundle of the normal bundle $\nu(M)$. The dimension over M of $\nu_0(M)$ is called the rank (see [O2, O3]). So, one has the more general question of whether a submanifold or higher rank and curvature normals of constant length must have constant principal curvatures (see [HOT]).

In this article, we show that only globally the questions above have a positive answer. Namely, if M is complete and it does not split off a curve.

Theorem 1.1 *Let M^n $(n \geq 2)$ be a connected, simple connected and complete riemannian manifold and let $f : M \to \mathbb{R}^N$ be an irreducible isometric immersion with $rank_f(M) \geq 1$ whose curvature normals are all of constant length. Assume furthermore, that either the number of curvature normals is constant on M or that $rank_f^{loc}(M) = rank_f(M)$. Then, any curvature normal field is globally defined and parallel in the normal connection.*

Let us recall that $rank_f^{loc}(M)$ is the maximum of $rank_f(U)$, where U runs over the open subsets of M.
It is interesting to note that this result is not true without the completeness assumption (see §5). It is also not true in hyperbolic space [W1]. The theorem above can also be formulated in terms of the adapted Gauss map $q \mapsto (\nu_0(M))_q$
For obtaining next corollaries we need first to generalize the main result in [CO, OW] to a global framework. This is not trivial at all and it is given in §3 .

Theorem 1.2 *Let M^n $(n \geq 2)$ be a connected, simple connected and complete riemannian manifold and let $f : M \to \mathbb{R}^N$ be an irreducible isometric immersion. If there exists a non-umbilical isoparametric parallel normal section then $f : M \to \mathbb{R}^N$ has constant principal curvatures.*

In order to prove last theorem, since we work in a global setting, we need to construct globally the parallel focal (immersed) manifold M_ξ. This is done in §2 where we use the so called bundle-like metrics [Re]

By combining the above theorems we obtain the following corollary.

Corollary 1.3 *Let M^n $(n \geq 2)$ be a connected, simple connected and complete riemannian manifold and let $f : M \to \mathbb{R}^N$ be an irreducible and full isometric* *immersion with $rank_f(M) \geq 2$ and curvature normals of constant length. Assume furthermore that either the number of curvature normals is constant on M or that $rank_f^{loc}(M) = rank_f(M)$. Then M is a submanifold with constant principal curvatures.*

This corollary generalize the main result in [O2]. It should be pointed out that most methods used in this paper differs from those used in this reference. Moreover, our arguments give a substantial simplification of the arguments in [O2], which at the same time give a better geometric understanding of the problem (we do not use homogeneity). This is also one of the goals of this paper.

If in addition, M has flat normal bundle we obtain the following corollary which provides a global characterization of isoparametric submanifolds of euclidean space.

Corollary 1.4 *Let M^n $(n \geq 2)$ be a connected, simple connected and complete riemannian manifold and let $f : M \to \mathbb{R}^N$ be an irreducible isometric immersion with flat normal bundle. Then the following properties are equivalent:*

(i) f is an isoparametric isometric immersion.

(ii) f has curvature normals of constant length (or equivalently, the distance to any focal hyperplane is constant).

(iii) The third fundamental form B has constant eigenvalues.

The result above holds locally for Kähler hypersurfaces of spaces of constant holomorphic sectional curvature with constant principal curvatures [St, Lemma 9, pp. 383]. The particular case of last corollary, when all curvature normals have the same (constant) length was proved (locally) in [N].

It is important to observe that the corollary above is not true for curves (curves with constant curvature are examples). Moreover, in §5 , we construct local counterexamples to Corollary 1.4 which show that the completeness assumptions cannot be dropped. These examples are deformations of the sphere $\mathbb{S}^N$ without two antipodal points, canonically immersed in the first factor of $\mathbb{R}^{N+1} \times \mathbb{R}^K$.

Note that part *(iii)* (or, equivalently, part *(ii)*) of Corollary 1.4 can be used for giving a global definition of isoparametricity in terms of the constancy of the eigenvalues of the third fundamental form $B = \sum_{j=1}^{N-n} A_{e_j}^2$, where A is the shape operator and $e_1, \cdots, e_{N-n}$ is an orthonormal basis of the normal space.

2 Preliminaries and basic facts

Curves are always assumed to be piece-wise differentiable and other maps are assumed to be differentiable. An isometric immersion $f : M \to \mathbb{R}^N$ is called *reducible* if $M = M_1 \times M_2$ and $f = f_1 \times f_2$, where $f_1 : M_1 \to \mathbb{R}^{N_1}$, $f_2 : M_2 \to \mathbb{R}^{N_2}$ and

$N = N_1 + N_2$. Local and global conditions for such a splitting are given by the so called Moore's Lemma (see [M], [BCO]). An isometric immersion $f : M \to \mathbb{R}^N$ is said to be $full$ if $f(M)$ is not contained in a proper affine subspace of $\mathbb{R}^N$. The normal bundle $\nu(M)$ will be regarded as $\nu(M) = \{(p, w) \in M \times \mathbb{R}^N : w \perp df(T_pM)\}$, where $df(T_pM)$ is identified to a linear subspace of $\mathbb{R}^N$. The fibres of the normal bundle will be often regarded as $\nu_p(M) = \{w \in \mathbb{R}^N : w \perp df(T_pM)\}$.

Let $f : M \to \mathbb{R}^N$ be an isometric immersion and let $\nu_0(M)$ be the subbundle of $\nu(M)$ which consists of the fixed points of the action of the restricted normal holonomy group, i.e., $\nu_0(M)$ is the maximal parallel (locally) flat subbundle of $\nu(M)$ (cf. [O2, O3]). The *rank* of the isometric immersion $f : M \to \mathbb{R}^n$ is the dimension (over M) of $\nu_0(M)$ and it is denoted by $rank_f(M)$. Let $\nu_0^{loc}(M)_p$ be the fixed points of the action of the local holonomy group at p. The local rank of f is defined by: $rank_f^{loc}(M) := max_{p \in M}\{dim(\nu_0^{loc}(M)_p)\}$. Note that $rank_f(M) \leq rank_f^{loc}(M)$ and the equality holds if and only if $\nu_0^{loc}(M)_p = \nu_0(M)_p$, for all $p \in M$. In other words, if M is simply connected, $rank_f(M) = rank_f^{loc}(M)$ if and only if any locally defined parallel normal field extends to a global one.

Assume that $rank_f(M) \geq 1$. By the Ricci identity the shape operators of sections of $\nu_0(M)$ commute and so they induce a decomposition $T_pM = E_1(p) \oplus \cdots \oplus E_{g(p)}(p)$ into simultaneous eigenspaces. Associated to this decomposition there are well defined curvature normals $\eta_i(p)$ such that $A_\xi|_{E_i(p)} = \lambda_i(\xi) Id = \langle \xi, \eta_i(p) \rangle Id_{E_i(p)}$ for all $\xi \in \nu_0(M)_p$. The curvature normals have the following continuity property which is standard to show:

(*) *Let $(p_m)_{m\in\mathbb{N}}$ be a sequence in M^n converging to p and let $(\eta_1(p_m), \cdots, \eta_n(p_m))$ be the curvature normals at p_m (chosen in any order and repeated according to the multiplicity). Then there exists a subsequence $(p_{m_j})_{j\in\mathbb{N}}$ such that $(\eta_1(p_{m_j}), \cdots, \eta_n(p_{m_j}))$ converges to the curvature normals $(\eta_1(p), \cdots, \eta_n(p))$ (in some order) at p .*

It is also standard to show that there exists an open and dense subset $\Omega \subset M$ where locally the number of eigendistributions is constant (or equivalently the number of distinct curvature normals is locally constant). In this case, the eigenspaces define, locally in Ω, C^∞ distributions and their associated curvature normals define C^∞ locally defined normal sections. It is standard to show, using the Codazzi identity, that any eigendistribution is integrable on Ω (in general the leaves are not totally umbilical, unless $\nu_0(M) = \nu(M)$). If $dim(E_i) \geq 2$, then $\nabla^\perp_X \eta_i = 0$ if X lies in E_i, as can be shown using the Codazzi identity of the shape operator. The $\nabla^\perp$-parallelism of η_i in the directions orthogonal to E_i is equivalent to the autoparallelism of E_i (this is also related to the Codazzi identity and a proof can be found in Lemma 2.3 of [O2]. Namely,

Lemma 2.1 *We are under the above notation an assumptions. Then*

(i) E_i is autoparallel if and only if $\nabla^\perp_Z \eta_i = 0$ for all Z lying in $E_i^\perp$.

(ii) If $dim(E_i) \geq 2$ then η_i is parallel if and only if E_i is autoparallel.

Remark 2.2 Observe that $R^{\perp}(X,Y)\xi = 0$ for all sections $\xi \in \nu_0(M)$. Then, by the Ricci identity $\langle R^{\perp}(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle$, A_ξ commutes with all shape operators. Then any eigendistribution E_i is invariant under all shape operators, or equivalently, $\alpha(E_i, E_j) = 0$, if $i \neq j$, where α is the second fundamental form.

Definition. The multiplicity of a curvature normal η_p at $p \in M$ is the dimension of its associated eigenspace.

Definition. A C^∞ locally defined normal section η is called a curvature normal field if it is a curvature normal at any point and the multiplicity is constant.

Remark 2.3 It is standard to show that:
(i) The associated eigendistribution to a curvature normal field is a C^∞ distribution.
(ii) If η_p is a curvature normal at $p \in M$ with multiplicity 1, then η_p extends locally to a curvature normal field.

Let L_f be the set of the lengths of the curvature normals at points of M i.e., $L_f := \{\|\eta_p\| : p \in M \text{ and } \eta_p \text{ is a principal curvature at } p\}$. We say that the isometric immersion has *curvature normal of constant lengths* if L_f is finite. Let $\xi_1(p), \ldots, \xi_r(p)$ be a orthonormal frame of $(\nu_0(M))_p$. It is standard to show that $B^0_p := \sum_i A^2_{\xi_i}$ does not depend on the frame $\xi_1(p), \ldots, \xi_r(p)$. Moreover, an isometric immersion has curvature normals of constant length if and only if B^0 has constant eigenvalues on M (i.e., not depending on $p \in M$). In fact,

$$B^0_{|E_i(p)} = \sum_{j=1}^{r} \langle \xi_j(p), \eta_i(p)\rangle^2 Id_{E_i(p)} = ||\eta_i(p)||^2 Id_{E_i(p)}$$

Let us define the *adapted Gauss maps* $G^0 : M \to Gr(r, N)$, from M into the Grassmannian of $r = rank_f(M)$ planes of $\mathbb{R}^N$, given by

$$p \mapsto (\nu_0(M))_p$$

In the case that M has flat normal bundle then the adapted Gauss map G^0 coincides with the usual Gauss map G. Let g^0 be the (possibly degenerated) metric in M induced by G^0. Then it is standard to show (cf [N]) that

$$g^0(X,Y) = \langle B^0(X), Y\rangle$$

If M has flat normal bundle then B^0 coincides with the so called third fundamental form B. Because of that we will refer to B^0 as the adapted third fundamental form (cf. [N]).

A section ξ of the normal bundle $\nu(M)$ is called *isoparametric* if the eigenvalues of the shape operator A_ξ are constant on M. Let $T_pM = E^\xi_1(p) \oplus \cdots \oplus E^\xi_{\tilde{g}(p)}(p)$ be the decomposition into eigenspaces of A_{ξ_p}. Since A_ξ has constant eigenvalues we have that $\tilde{g}(p)$ does not depend on p. So, the eigenspaces define C^∞ distributions on M. If ξ is parallel then any eigendistribution E^{ξ} is autoparallel (by Codazzi identity). So, the leaves of any eigendistribution are totally geodesic submanifolds of M. Assume that ξ is a parallel isoparametric normal section and that 1 is an eigenvalue of ξ. Let $E := ker(Id - A_{\xi})$ be the eigendistribution of the eigenvalue 1 and let $\mathcal{H} := E^{\perp}$ be the so called *horizontal* distribution. From Remark 2.2 we have that $\mathcal{H}$ as well as E are invariant by all shape operators. The leaves of E are the connected components of the fibers of the (constant rank) map $p \xrightarrow{f_{\xi}} f(p) + \xi(p)$ from M to $\mathbb{R}^N$. So, the involutive differential system given by E is regular in the sense of Palais [P]. Let $\pi : M \to M_{\xi}$ be the projection on the quotient manifold given by the identification of points on the same leaf of the eigendistribution E (see [P]). We have the following proposition (cf. [CR], [R]).

Proposition 2.4 *In the above situation assume also that M is complete. Then,*

(i) *M_{ξ} is a differentiable Hausdorff manifold and $\pi : M \to M_{\xi}$ is a submersion.*

(ii) *There exist an immersion $\tilde{f}_{\xi} : M_{\xi} \to \mathbb{R}^N$ such that $f_{\xi} = \tilde{f}_{\xi} \circ \pi$*

(iii) *M_{ξ} is a complete riemannian manifold with respect to the riemannian metric induced by $\tilde{f}_{\xi}$.*

Proof. In order to prove that M_{ξ} is Hausdorff we define a new riemannian metric $\langle , \rangle_{\xi}$ in M given by $\langle X, X\rangle_{\xi} := \langle (Id - A_{\xi})X, (Id - A_{\xi})X\rangle$ for $X \in \mathcal{H}$, $\langle X, X\rangle_{\xi} := \langle X, X\rangle$ for $X \in E$ and $\langle E, \mathcal{H}\rangle_{\xi} = 0$. Since the eigenvalues of A_{ξ} are constant (and hence bounded) then there exist constant $c, C > 0$ such that $c\langle , \rangle \leq \langle , \rangle_{\xi} \leq C\langle , \rangle$. So, $(M, \langle , \rangle_{\xi})$ is also a complete riemannian manifold. Observe that f_{ξ} is locally a riemannian submersion from $(M, \langle , \rangle_{\xi})$ on its image which is locally a parallel focal manifold to M. So, this new metric is a bundle-like metric (in the sense of [Re]) with respect to the foliation given by the leaves of the distribution E (Notice that E is also autoparallel with respect to the new metric $\langle , \rangle_{\xi}$ as it can be shown using the well-known Koszul identity for the covariant derivative). Now, part (i) is a direct consequences of Corollary 3 of [Re, pp.129]. Part (ii) is standard to show. For part (iii) one has to observe that π is a riemannian submersion from $(M, \langle , \rangle_{\xi})$ into M_{ξ}, where M_{ξ} has the metric induced by $\tilde{f}_{\xi}$. Then M_{ξ} is also complete. □

Remark 2.5 Notice that $\pi : (M, \langle , \rangle_{\xi}) \to M_{\xi}$ is a riemannian submersion. So, by [H], it is a fiber bundle and therefore any piecewise differentiable curve $c : [0,1] \to M_{\xi}$ can be lifted to a horizontal curve $\tilde{c}_p(t)$, starting at any given point $p \in \pi^{-1}(c(0))$. Moreover, since the fibers of π are totally geodesic submanifolds of M it follows that $\pi : (M, \langle , \rangle_{\xi}) \to M_{\xi}$ is a fiber bundle with structure group the isometry group of a fixed fibre [H].
Observe also that if M is simply connected then M_{ξ} is so, since the fibers of $\pi : M \to M_{\xi}$ are connected.

Note that $df(\mathcal{H}_p) = d\tilde{f}_\xi(T_{\pi(p)}(M_\xi))$ and so, $\nu_p(M) \subset \nu_{\pi(p)}(M_\xi)$, for all $p \in M$. Let $i : M \to \nu(M_\xi)$ be the immersion defined by

$$i(p) = f(p) - f_\xi(p) = -\xi(p) \ \in \nu_{\pi(p)}(M_\xi)$$

Observe that $o_\nu \circ i = \pi$, where $o_\nu : \nu(M_\xi) \to M_\xi$ is the projection to the base. So, for verifying that i is an immersion, we must only show that $di_{|E(p)}$ is injective for all $p \in M$. In fact, if $c(t)$ is a curve in $\pi^{-1}(\pi(p))$ with $c(0) = p$, then $i \circ c(t)$ is a curve in $\nu_{\pi(p)}(M_\xi)$. We have that

$$\frac{d}{dt}|_0 \ i(c(t)) = -\frac{d}{dt}|_0 \ \xi(c(t)) = df(A_{\xi(p)}c'(0)) = df(c'(0))$$

since $A_{\xi(p)|E(p)} = Id_{E(p)}$. Then $di(c'(0)) = df(c'(0))$ and so i is an immersion. The following commutative diagram gives a picture of the maps we are dealing with.

$$\begin{array}{ccccc} & & M & & \\ & \overset{i}{\swarrow} & \downarrow \pi & \overset{f_\xi}{\searrow} & \\ \nu(M_\xi) & \xrightarrow{o_\nu} & M_\xi & \xrightarrow{\tilde{f}_\xi} & \mathbb{R}^N \end{array}$$

Let $\beta(t)$ be a horizontal curve in M (with respect to the horizontal distribution $\mathcal{H} = E^\perp$). Then

$$\frac{d}{dt} \ \xi(\beta(t)) = -\, df(A_{\xi(\beta(t))}\beta'(t)) \subset df(\mathcal{H}_{\beta(t)}) = d\tilde{f}_\xi(T_{\pi(\beta(t))}(M_\xi))$$

since $\mathcal{H}$ is invariant by the shape operators. Then $-\,\xi(\beta(t))$ may be regarded as a parallel normal field to M_ξ along $\pi(\beta(t))$ (cf. [HOT]).
Let, for $c : [0,1] \to M_\xi$, $\tilde{c}_q(t)$ denote the horizontal lift of c to M with $\tilde{c}_q(0) = q \in \pi^{-1}(\pi(q))$. Then, by what observed before we have (cf. [HOT])

$$(I) \qquad\qquad i \circ \tilde{c}_q(1) = \tau_c^\perp(i(q))$$

Assumption: in the rest of this article, unless otherwise stated, M (and hence M_ξ) is assumed to be simply connected and complete.

For any given $\bar{p} \in M_\xi$ it is defined the holonomy group $\Phi_{\bar{p}}$ at $\bar{p}$. Namely,

$$\Phi_{\bar{p}} = \{\tau_c^{\mathcal{H}} : c \text{ is a loop in } M_\xi \text{ based at } \bar{p}\}$$

where $\tau_c^{\mathcal{H}}$ is the isometry of $\pi^{-1}(\bar{p})$ defined by $\tau_c^{\mathcal{H}}(q) = \tilde{c}_q(1)$. The holonomy group $\Phi_{\bar{p}}$ is connected since M_ξ is simply connected (see Remark 2.5). We should point out that for an arbitrary riemannian submersion from a complete riemannian manifold, with non-totally geodesic fibres, the holonomy group is not in general a finite dimensional Lie group. As in the case of homogeneous vector bundles, M is foliated by the so called holonomy subbundles (this is indeed a singular foliation). Namely, if $u \in M$, then the holonomy subbundle $Hol_u^{\mathcal{H}}(M_\xi)$ through u consists of those points in M which can be joint to u by a horizontal curve. Let now $\Phi_{\bar{p}}^{\perp}$ be the normal holonomy group of $\tilde{f} : M_\xi \to \mathbb{R}^N$ at $\bar{p} \in M_\xi$ and let, for $v \in \nu(M)$, $Hol_v^{\perp}(M_\xi)$ be the holonomy subbundle of $\nu(M)$ through v (see [HOT]). Then by the equality (I) we have that

$$(II) \qquad\qquad i(Hol_u^{\mathcal{H}}(M_\xi)) = Hol_{i(u)}^{\perp}(M_\xi)$$

Remark 2.6 Normal holonomy groups, for riemannian submanifolds of euclidean or lorentzian space, act polarly on normal spaces (see Theorem 2.8 in [OW]). So, equalities $(I), (II)$ imply that $\Phi_{\bar{p}}$ acts (locally) polarly on the (totally geodesic fibre) $\pi^{-1}(\bar{p})$ (see Lemma 2.6 of [OW]). Hence, normal spaces to maximal dimensional orbits of $\Phi_{\bar{p}}$ define an autoparallel distribution on an open and dense subset of the fibre.

We finish this section with a criterion which implies that the number of curvature normals is independent of $p \in M$.

Lemma 2.7 *Let $f : M \to \mathbb{R}^N$ be an isometric immersion with $rank_f(M) \geq 1$, where M is connected, and let Ω be the open and dense subset of M where the number of curvature normals is locally constant (and so the curvature normals define C^∞ normal sections). Assume that on each connected component of Ω the scalar product of any two arbitrary given curvature normals is constant. Then the number of curvature normals is constant on M.*

Proof. Let us consider the symmetric endomorphism $\tilde{B}^0$ of $TM \otimes TM$ defined by

$$\tilde{B}_p^0(v \otimes w) = \sum_{s=1}^{r} A_{\xi_s^p} v \otimes A_{\xi_s^p} w$$

where A is the shape operator and $\xi_1^p, \cdots, \xi_r^p$ is an orthonormal basis of $\nu_0(M)_p$ and $v, w \in T_pM$. Let now $p \in \Omega$ and let $v_i, v_j \neq 0$ be two common eigenvectors of the family of shape operators, restricted to $\nu_0(M)_p$, corresponding to the eigenvalue function $\langle \eta_i(p), \ \rangle$. Then

$$\tilde{B}_p^0(v_i \otimes v_j) = \sum_{s=1}^{k} \langle \eta_i(p), \xi_s^p \rangle \langle \eta_j(p), \xi_s^p \rangle v_i \otimes v_j = \langle \eta_i(p), \eta_j(p) \rangle v_i \otimes v_j$$

Then the eigenvalues of $\tilde{B}_p^0$ are $\langle \eta_i(p), \eta_j(p) \rangle$, $i, j = 1, \cdots, n = dim(M)$ (here, in contrast with the rest of the article, the curvatures normals are repeated according to the multiplicity and so it may happen that $\eta_i(p) = \eta_j(p)$ with $i \neq j$). We have, from the assumptions, that the eigenvalues of $\tilde{B}^0$ are locally constant on the dense

subset Ω of M. Then the eigenvalues of $\tilde{B}^0$ are constant on M. In fact, the coefficients of the characteristic polynomial of $\tilde{B}^0$ must be constant since they are locally constant in Ω.
Now observe the curvature normals $\eta_1(p), \cdots, \eta_n(p)$ are completely determined, up to orthogonal transformations, by the numbers $\langle \eta_i(p), \eta_j(p) \rangle$ which are the eigenvalues of $\tilde{B}^0_p$. So, if we know the eigenvalues of $\tilde{B}^0_p$ there are only finitely many possibilities, up to orthogonal transformations, for choosing $\eta_1(p), \cdots, \eta_n(p)$. Then there exists $V_1, \cdots, V_k$ disjoint open subsets of Ω such that $\Omega = \cup_{i=1}^k V_i$ and with the property that the curvature normals $\eta_1(p), \cdots, \eta_n(p)$ are orthogonally equivalent to $\eta_1(q), \cdots, \eta_n(q)$ if and only if $p, q \in V_i$, for some $i = 1, \cdots, r$. From the continuity property *(*)* of the curvatures normals we have that if x belongs to the closure $\bar{V}_i$ of V_i then the curvatures normal $\eta_1(x), \cdots, \eta_n(x)$ are orthogonal equivalent to the curvatures normals at any point of V_i. This implies that $\bar{V}_i$ is disjoint to $\bar{V}_j$ if $i \neq j$. But $M = \cup_{i=1}^k \bar{V}_i$, since Ω is dense. This is contradiction, since M is connected, unless $k = 1$. This implies the lemma. □

3 Global isoparametric parallel normal sections

We keep the notation of the previous sections. Let $f : M \to \mathbb{R}^N$ be a simply connected and complete (immersed) submanifold and let ξ be a parallel isoparametric normal section. Let $\lambda_1, \cdots, \lambda_g$ be the distinct non-zero eigenvalues of the shape operator A_ξ with associated autoparallel eigendistributions $E_0, \cdots, E_g$ (E_0 is the eigendistribution associated with 0 and may be trivial).
The object of this section is to prove Theorem 1.2 which is a global version of the main theorems in [CO, OW]. Since there exist simply connected and complete riemannian manifolds which are locally reducible at any point, we cannot expect to apply directly the local arguments given in [OW]. Mainly, because in this last reference the local irreducibility is assumed at any point. The key fact for the global setting is Lemma 3.1 which implies that the holonomy group $\Phi_{\bar{p}}$, associated to any submersion $M \overset{\pi_i}{\to} M_{\lambda^{-1}\xi}$ (with totally geodesic fibers) have no singular orbits in $\pi_i^{-1}(\bar{p})$ and so the normal spaces to such orbits define a global C^∞ distribution.
Let us state the following general lemma that we need.

Lemma 3.1 *Let G be a Lie group which acts by isometries and locally polar on a riemannian manifold M. Assume that the normal bundle of the principal G-orbits define a parallel distribution $\tilde{\nu}$ of the open and dense set U on which principal orbits are defined. Then, all G-orbits have the same dimension and so, $\tilde{\nu}$ extend to a global parallel distribution on M.*

Proof. Fix an arbitrary point $p \in M$. Let X be a Killing field of G which belong to the Lie algebra of the isotropy group G_p. Let v_p be a principal (normal) vector of the action of the isotropy group G_p on $\nu(G.p)$. It is well known that for small t the G-orbits through $exp(t.v_p)$ are principal orbits of the G action. Let $Exp(sX)$ be the monoparametric subgroup generated by the Killing field. Define $f(s,t) := Exp(tX).exp(sv_p)$. Observe that $\frac{\partial}{\partial s}f$ is parallel along the s parameter because it is the tangent vector of the respective geodesic. Also, by hypothesis $\frac{\partial}{\partial s}f$ is parallel along the t parameter because the normal space of principal orbit moves parallerly on M. So, $0 = \frac{D}{\partial t}\frac{\partial}{\partial s}f|_{s=0} = \frac{d}{dt}(Exp(tX)_*.v_p)$ which implies that the connected component of the identity of G_p acts trivially on $\nu(G.p)$. Thus, all G-orbits has the same dimension and now it is standard to show that $\tilde{\nu}$ extend to a parallel distribution on M. $\square$

Proof of Theorem 1.2. We will only sketch how to adapt the arguments in [OW] to our situation. The proof of this theorem, after giving a global version of Theorem 3.3 in [OW], will follow exactly in the same way as for the local one. First of all we remark that we should regard, as in [OW], the euclidean space as a horosphere of hyperbolic space (regarded as a riemannian hypersurface of lorentzian space). This is for dealing with the eigendistribution $E_0 = ker(A_\xi)$. In particular, we should have stated Proposition 2.4 for submanifolds of lorentzian space contained in hyperbolic space. But, for the sake of clearness, we will only refer to euclidean submanifolds because the proofs are essentially the same.
For the global version of Theorem 3.3 of [OW] we have to replace the normal holonomy group of the focal parallel manifold M_ξ by the holonomy group of the submersion $M \xrightarrow{\pi} M_\xi$. So, the holonomy tube around the focal manifold, through any given point u of M, has to be replaced by the holonomy subbundles $Hol_u^{\mathcal{H}}(M_\xi)$. Both constructions are locally the same due to equalities $(I), (II)$ of the previous section. It is defined a global, a priori singular, vertical foliation $\tilde{\nu}$ on M given by the normal spaces to the holonomy subbundles (or equivalently, by the normal spaces to orbits of holonomy groups). In an open and dense subset Ω of M,where the dimension of $\tilde{\nu}$ is maximal, $\tilde{\nu}$ is a C^∞ distribution. (Observe that the holonomy group $\Phi_{\pi(u)}$ acts by isometries on the complete riemannian manifold $\pi^{-1}(\pi(u))$ and so, maximal dimensional orbits are dense. Moreover, by Remark 2.6, this action is locally polar). By the local arguments in [OW] we have that $\tilde{\nu}$ is not only autoparallel but also a parallel distribution in Ω. Since this distribution, when restricted to any fibre $\pi^{-1}(\pi(u))$, which is totally geodesic, coincides with the distribution given by the normal spaces to the orbits of the holonomy group $\Phi_{\pi(u)}$. We apply now Lemma 3.1 to conclude that all the orbits of $\Phi_{\pi(u)}$ have the same dimension and so $\tilde{\nu}$ is a C^∞ distribution in M which must be also parallel. By the local arguments in [OW] we have that $\alpha(\tilde{\nu}, \tilde{\nu}^\perp) = 0$ on Ω and hence, by continuity, on M. Then $f : M \to M_\xi$ would split unless $\tilde{\nu}$ is trivial. $\square$.

Remark 3.2

i) Let $f : M^n \to \mathbb{R}^N$, $n \geq 2$, be a full isometric immersion with $rank_f(M) \geq 2$. Then it admits a non-umbilical parallel normal section. In fact, if ξ is an umbilical parallel normal section, then $A_\xi = \lambda Id$, where λ is constant due to the Codazzi identity. Let η be another umbilical parallel normal section which is not a multiple of ξ. Then there exists a (constant) linear combination $\mu \neq 0$ of ξ and η and such that $A\mu = 0$. Then M reduces codimension, since μ must be constant on M since the tangent and the normal part of its euclidean derivative are both zero.

(ii) Let $f : M \to \mathbb{R}^N$ be a submanifold with constant principal curvatures. Then M is a parallel focal manifold to an isoparametric submanifold [HOT]. So, if the immersion is irreducible then $f(M)$ lies in a sphere (since an isoparametric submanifold is the product of a euclidean space by an isoparametric submanifold contained in a sphere)

4 Isometric immersions with curvature normal fields of constant length

Let $f : M^n \to \mathbb{R}^N$ $(n \geq 2)$ be a full and irreducible isometric immersion, where M is a complete and simply connected riemannian manifold ($n \geq 2$). Assume that $rank_f(M) = r \geq 1$ and that the submanifold M has curvature normals of constant length. This is equivalent to the fact that the tensor B^0 has constant eigenvalues, where $B^0_p = \sum_{i=1}^r A^2_{\xi_i(p)}$ and $\xi_1(p), \cdots, \xi_r(p)$ is an orthonormal basis of $\nu_0(M)_p$ (see section 2).

Assume that the number $g(p)$ of curvature normals does not depend on $p \in M$. Note that this implies that the curvature normals are globally defined. In fact, the set

$$C = \{\eta_p : p \in M \text{ and } \eta_p \text{ is a curvature normal at } p\}$$

is a submanifold of the normal space $\nu(M)$ with dimension n. Moreover, the fact that g does not depend on p implies that the projection of the normal bundle to the base, when restricted to C, is a covering map. Since M is simple connected C must have exactly g connected components given by the images of globally defined curvature normals $\eta_1, \ldots, \eta_g$. Observe that this implies that $dim(E_i)$ is constant on M for any i. Without loss of generality we will assume that $\|\eta_i\| \geq \|\eta_j\|$ if $i \leq j$.

If $\eta_1, \cdots, \eta_g$ are all parallel normal sections, then $f(M)$ is always contained in a sphere and it has constant principal curvatures if $g \geq 2$ (see Theorem 1.2 and Remark 3.2 *(ii)*).

Let us then assume that the curvature normal are not all parallel. Let $k \in \{1, \cdots, g\}$ be the first index such that η_k is not parallel (i.e. η_k is (one of) the longest non-parallel curvature normal). Let, for $i, j \in \{1, \cdots, g\}$, $h_{ij} : M \to \mathbb{R}$ be defined by

$$h_{ij} = \langle \eta_j, \eta_j \rangle - \langle \eta_i, \eta_j \rangle = \langle \eta_j - \eta_i, \eta_j \rangle$$

By the Cauchy-Schwartz inequality we have that

$$h_{ij} > 0 \qquad \text{if } i > j$$

Let J be an arbitrary subset of $\{1, \cdots, g\} - \{k\}$ and let

$$\Omega_J = (\{p \in M : h_{jk}(p) = 0 \Leftrightarrow j \in J\})^o$$

where $(\ )^o$ denotes the interior. Observe that $\Omega_J = \emptyset$ if J is not contained in $\{1, \cdots, k-1\}$. Notice also that $\Omega_\emptyset = (\{p \in M : h_{jk}(p) \neq 0 \ \forall\, j = 1, \cdots, k-1, k+1, \cdots, g\})^o$. In particular $\Omega_\emptyset = M$, if $k = 1$. It is standard to show that

$$\Omega = \bigcup_{J \subset \{1, \cdots, k-1\}} \Omega_J$$

is an open and dense subset of M.
We will show that the eigendistribution E_k associated with the curvature normal η_k is autoparallel. It suffices to show that the restriction $E_k|_{\Omega_J}$ is autoparallel, for any $J \subset \{1, \cdots, k-1\}$. For this we will follow the ideas of section 2 of [O2]. Let $J \subset \{1, \cdots, k-1\}$. Without loss of generality we may assume that $J = \{1, \cdots, s\}$, where $s < k$. Observe that

(i) $\langle \eta_i, \eta_j \rangle$ is constant if $i, j < k$ (in particular, if $i, j \leq s$), since η_i, η_j are parallel.
(ii) $\langle \eta_i, \eta_i \rangle > \langle \eta_i, \eta_k \rangle$ for any $i \leq s$, since $\|\eta_i\| \geq \|\eta_k\|$.
(iii) $\langle \eta_k, \eta_k \rangle > \langle \eta_l, \eta_k \rangle$ for any $l > k$, since $\|\eta_k\| \geq \|\eta_l\|$.
(iv) $\langle \eta_i, \eta_k \rangle$ for $i \leq s$ is constant in Ω_J, since $\langle \eta_i, \eta_k \rangle = \|\eta_k\|^2$ and the curvature normals have constant length.

Let $p \in \Omega_J$ and let ξ^p be the parallel normal section such that $\xi^p(p) = \eta_k(p)$. The shape operator A_{ξ^p} does not distinguish, near p, the eigendistribution E_k, unless $J = \emptyset$. This is because, from the definition of Ω_J, $\lambda_k(\xi^p(p)) - \lambda_i(\xi^p(p)) = h_{ik}(p) = 0$, for $i = 1, \cdots, s$. Also note that in Ω_J we have that $\lambda_k(\xi^p(p)) - \lambda_l(\xi^p(p)) = h_{lk}(p) \neq 0$, for all $k \neq l > s$. It is standard to show, using *(i), (ii) (iii)* and *(iv)* that there exists a parallel normal section $\bar\xi$, which is a linear combination of $\eta_1, \cdots, \eta_s$, and such that $\bar\xi + \xi^p$ distinguish, at p and so near p, the eigendistribution E_k (i.e. $\lambda_j(\bar\xi(p) + \xi^p(p)) \neq \lambda_k(\bar\xi(p) + \xi^p(p))$ if $j \neq k$). (Observe that we have, from *(iv)*, that $\langle \bar\xi, \eta_k \rangle = c$ is a constant). We have that $\langle \bar\xi + \xi^p, \eta_k \rangle = c + \langle \xi^p, \eta_k \rangle$. Using the Cauchy-Schwartz inequality, since $\|\xi^p\| = \|\eta_k\|$ is constant, we obtain that the function $\langle \xi^p, \eta_k \rangle$ has a maximum at p and hence $\langle \bar\xi + \xi^p, \eta_k \rangle$ achieves its maximum at p. Hence its differential is zero at p. Then the symmetric tensor

$$T^p = A_{\bar\xi + \xi^p} - \langle \bar\xi + \xi^p, \eta_k \rangle Id$$

satisfies the Codazzi identity (only) at p, since $A_{\bar\xi + \xi^p}$ and Id satisfies the Codazzi identity. Namely, $\nabla_X(T^p)(Y) = \nabla_Y(T^p)(X)$, where ∇ is the Levi-Civita connection of M and $\nabla_X(T^p)(Y) = \nabla_X T^p(Y) - T^p(\nabla_X Y)$. Equivalently, the tensor $\langle \nabla_X(T^p)(Y), Z \rangle$ is symmetric in all its three variables. Since $E_k = ker(T^p)$ near p we have that E_k is autoparallel at p. In fact, if X, Y are tangent fields which lie in E_k and Z is arbitrary then

$$\langle \nabla_X(T^p)(Y), Z \rangle_p = \langle \nabla_X T^p(Y) - T^p(\nabla_X Y), Z \rangle_p = -\langle T^p(\nabla_X Y), Z \rangle_p$$

By the Codazzi identity we have that

$$-\langle T^p(\nabla_X Y), Z\rangle_p = \langle \nabla_Z(T^p)(Y), X\rangle_p = \langle \nabla_Z T^p(Y) - T^p(\nabla_Z Y), X\rangle_p =$$
$$-\langle T^p(\nabla_Z Y), X\rangle_p = -\langle \nabla_Z Y, T^p(X)\rangle_p = 0$$

Then $-\langle T^p(\nabla_X Y), Z\rangle_p = 0$ for Z arbitrary. Then, $(\nabla_X Y)_p$ belongs to $(ker(T^p))_p = E_k(p)$. Since p is arbitrary we conclude that E_k is autoparallel in Ω_J. Since J is arbitrary we conclude that E_k is an autoparallel distribution in M, since Ω is dense. Applying Lemma 2.1 we must have that $dim(E_k) = 1$, since η_k is not parallel. We have also that $E_k^\perp = E_1 \oplus \cdots \oplus E_{k-1} \oplus E_{k+1} \oplus \cdots \oplus E_g$ is integrable. Namely,

Lemma 4.1 *Let $f : M^n \to \mathbb{R}^N$ be a full isometric immersion, where M is a complete and simply connected riemannian manifold. Assume that $rank_f(M) = r \geq 1$ and that the submanifold M has C^∞ globally defined distinct curvatures normals $\eta_1, \cdots, \eta_g$ of constant length. Assume that η_k is not parallel (at some point) and that η_k is (one of) the longest with this property (i.e. $\|\eta_k\| \geq \|\eta_i\|$ if $\nabla^\perp \eta_i \neq 0$). Then*

(i) E_k is autoparallel and $dim(E_k) = 1$,

(ii) $E_k^\perp$ is integrable.

Proof. Part *(i)* was proved before. With respect to part *(ii)* we point out that it does not follow from the local arguments in [O2, pp. 611], since η_k could be parallel in an open subset. For those points we need to do global arguments which depends on the completeness of M.
Assume that η_k is not parallel at p. Then, by part *(i)* and Lemma 2.1, we have that $\nabla^\perp_{Z_p} \eta_k = 0$ if and only if $Z_p \in E_k^\perp(p)$. Let now X, Y be tangent fields lying in $E_k^\perp$. Since $\nu_0(M)$ is flat we have that $0 = R^\perp(X,Y)\eta_k = [\nabla^\perp_X, \nabla^\perp_Y]\eta_k - \nabla^\perp_{[X,Y]}\eta_k = -\nabla^\perp_{[X,Y]}\eta_k$. Then $[X,Y]_p \in E_k^\perp(p)$ and so $E_k^\perp$ is involutive at p. Notice that the set of points q in M such that $E_k^\perp$ is involutive at q is a closed subset of M. This implies that if $E_k^\perp$ is not involutive at some $p \in M$, then η_k is parallel in a neighbourhood of p. Let us assume that E_k is not involutive at some fixed p and let $\gamma : \mathbb{R} \to M$ be the unit speed geodesic in M with $\gamma(0) = p$ and $\gamma'(0) \in E_k(p)$ (i.e. γ is an integral manifold of the one dimensional autoparallel distribution E_k). We will show that η_k in not only parallel near p but also near $\gamma(t)$, for any $t \in \mathbb{R}$. Since M is simple connected and E_k has dimension 1 we can find a global defined unit vector field X such that $E_k = \mathbb{R}X$ and $X(p) = \gamma'(0)$. Let F_t be the flow of X on M, i.e. $F_t(p) = exp(tX_p)$. Observe that F_t is complete because M is complete. We claim that $E_k^\perp$ is invariant by F_t. In fact, let $w_p \in ({E_k}^\perp)_p$ we will show that $(F_t)_*(w_p) \in ({E_k}^\perp)_{F_t(p)}$. Let $r(s)$ be a curve with $w_p = r'(0)$ and let $J(t) := \frac{\partial}{\partial s} F_t(r(s))|_{s=0}$ be the Jacobi field along $F_t(p)$. Note that $J(0) = w_p$ and $\frac{D}{\partial t} J|_{t=0} = \nabla_{w_p} X \perp X$ (since $\|X\| = 1$). So, the Jacobi equation implies

that $J(t) \perp X(F_t(p))$ for all t. This implies that E_k is invariant by F_t because $J(1) = (F_t)_*(w_p)$. Since $F_t(p) = \gamma(t)$ we conclude that $E_k^\perp$ is not involutive at $\gamma(t)$, for any $t \in \mathbb{R}$. Then η_k is parallel in a neighbourhood of $\gamma(t)$. Using this information we will show that η_k is globally parallel, a contradiction.

Sublemma 4.2 *Under the above assumptions let $q \in M$ be arbitrary and let $c : [0,1] \to M$ be a curve with $c(0) = p$ and $c(1) = q$ (p as above) Let $\tau_c^\perp$ be the normal parallel transport along c. Then there exist a (piecewise smooth) horizontal curve (i.e. $h'(t)$ lies in $E_k^\perp$) $h : [0,1] \to M$ ending at $h(1) = q$ and a vertical geodesic $\gamma : [0,b] \to M$ (i.e. $\gamma'(t)$ lies in E_k) which starts at $\gamma(0) = p$ and ends at $\gamma(b) = h(0)$ and such that*

$$\tau_c^\perp = \tau_h^\perp \circ \tau_\gamma^\perp$$

Proof of Sublemma 4.2. Since, $R^\perp(E_k, {E_k}^\perp) = 0$, it is not hard to show that there exists a partition $0 = t_0 < t_1 < \cdots < t_d = 1$ such that, for any $i = 0, \cdots, d$, the restriction $c_{|[t_i,t_{i+1}]}$ is under the hypothesis of the lemma in [O1, appendix]. That is, there exist a horizontal curve h_i and a vertical curve γ_i, which we may assume that it is a (unit speed) geodesic, since E_k is totally geodesic, such that $\tau^\perp_{c_{|[t_{i-1},t_i]}} = \tau^\perp_{\gamma_i} \circ \tau^\perp_{h_i}$. Then

$$\tau_c^\perp = (\tau^\perp_{\gamma_d} \circ \tau^\perp_{h_d}) \circ \cdots \circ (\tau^\perp_{\gamma_1} \circ \tau^\perp_{h_1})$$

For simplicity we assume that $d = 2$. The general case can be obtained by iterating the procedure below. We have that $\gamma_2(t) = F_t(\gamma_2(0))$ or $\gamma_2(t) = F_{-t}(\gamma_2(0))$, depending on whether $\gamma_2'(0) = \pm X(\gamma_2(0))$, $t \in [0, a_2]$. We may assume $\gamma_2'(t) = X(\gamma_2(t))$ (otherwise, we change X by $-X$). For any fixed t we have that $s \mapsto \phi(s,t) := F_t(h_2(s))$ is a horizontal curve since the flow of X preserves the horizontal distribution $E_k^\perp$. Then, since $R^\perp(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t}) = 0$, we obtain that

$$\tau^\perp_{\gamma_2} \circ \tau^\perp_{h_2} = \tau^\perp_{\tilde{h}_2} \circ \tau^\perp_{\tilde{\gamma}_2}$$

where $\tilde{h}_2 = F_{a_2} \circ h_2$ and $\tilde{\gamma}_2(t) = F_t(h_2(0))$. Then

$$\tau_c^\perp = \tau^\perp_{\gamma_2} \circ \tau^\perp_{h_2} \circ \tau^\perp_{\gamma_1} \circ \tau^\perp_{h_1} = \tau^\perp_{\tilde{h}_2} \circ \tau^\perp_{\tilde{\gamma}_2} \circ \tau^\perp_{\gamma_1} \circ \tau^\perp_{h_1} = \tau^\perp_{\tilde{h}_2} \circ \tau^\perp_{\bar{\gamma}} \circ \tau^\perp_{h_1}$$

where $\bar{\gamma} = \gamma_1 * \tilde{\gamma}_2$ is a smooth geodesic obtained by gluing together γ_1 and $\tilde{\gamma}_2$ (recall that both geodesics are integral curves of X). Applying the same arguments to h_1 and $\bar{\gamma}$ we obtain that $\tau^\perp_{\bar{\gamma}} \circ \tau^\perp_{h_1} = \tau^\perp_{\tilde{h}_1} \circ \tau^\perp_{\gamma}$. So,

$$\tau_c^\perp = \tau_h^\perp \circ \tau_\gamma^\perp$$

where $h = \tilde{h}_1 * \tilde{h}_2$ is the (piece-wise smooth) horizontal curve obtained from running along $\tilde{h}_1$ and then along $\tilde{h}_2$ $\square$.

We continue with the proof of Lemma 4.1. Let $c : [0, 1] \to M$ be an arbitrary curve in M starting at p and such that E_k in not involutive at p. Let γ and h be the vertical and horizontal curves given by the Sublemma. We have shown that η_k is parallel along γ. By part (i) of this lemma and part (i) of Lemma 2.1 we have that η_k is also parallel along h. Then, $\eta_k(c(1)) = \tau_c^{\perp}(\eta_k(p))$. Then η_k is (globally) parallel. A contradiction □

Remark 4.3 Observe that we can replace, in the sublemma, the horizontal curve by a vertical one and vice versa (using the flow of X, as in the proof).

Remark 4.4 It is interesting to note that there exist manifolds M such that: there is a global vector field X on M, with a complementary distribution $\mathcal{D}$ which is invariant by the flow of X. Moreover, $\mathcal{D}$ is integrable on some open subsets of M but not integrable on the whole M. For example take $X = \frac{\partial}{\partial z}$ in $\mathbb{R}^3$ and $\mathcal{D} = ker(\alpha)$, where $\alpha = \phi(y)dx + dz$ and $\phi(y) := e^{-1/y}$ if $y > 0$ and $\phi(y) := 0$ if $y \leq 0$. It is clear that $\mathcal{D}$ is integrable on the open subset $\{y < 0\}$. Also, we have $d\alpha \wedge \alpha = \phi'(y)dy \wedge dx \wedge dz \neq 0$ if $y > 0$, which shows that $\mathcal{D}$ is not integrable on $\{y > 0\}$.
This illustrates why the proof of the integrability of $E_k^{\perp}$ does not follow easily from the integrability on an open subset.

Remark 4.5 A curvature normal is called *Dupin* if it is $\nabla^{\perp}$-parallel along the leaves of its associated eigendistribution (see [C]). Observe that the above lemma gives the proof of Theorem 1.1 under the additional Dupin hypothesis on the curvature normals.

We will need a local version of Lemma 4.1 for the last part of the proof of Theorem 1.1.

Lemma 4.6 *Let $f : M^n \to \mathbb{R}^N$ be a full isometric immersion, where M is a (non necessarily complete) riemannian manifold. Assume that $rank_f(M) = r \geq 1$ and that the submanifold M has (exactly) g distinct curvatures normals $\eta_1, \cdots, \eta_g$ of constant length which are globally defined and C^∞. Assume that η_k is not parallel at any point and that η_k has the following property for all $p \in M$: if $\langle \eta_j(p), \eta_k(p) \rangle = \|\eta_k\|^2$ (this is only possible if $\|\eta_j\|^2 > \|\eta_k\|^2$) then η_j is parallel in a neighbourhood of p. Then*

(i) E_k is autoparallel and $dim(E_k) = 1$,

(ii) $E_k^{\perp}$ is integrable.

Proof. Is just the same as in the local part of the proof of Lemma 4.1.

Let $\mathbf{L}_p$ be the leaf through $p \in M$ of the distribution $E_k^{\perp}$. Define $\tilde{\gamma}_p(t) := f(\gamma_p(t)) - f(p)$, where $\gamma_p(t) := F_t(p)$, F_t is the flow of the unit field X which spans the vertical distribution E_k and f is the immersion (see the proof of Lemma 4.1). Observe that $\gamma_p(t)$ is a geodesic in M, since E_k is autoparallel. We have the following lemma.

Lemma 4.7 *In the assumptions of Lemma 4.1, we have that:*

(i) $df(T_p(\mathbf{L}_p)) = df(T_{\gamma_p(t)}(\mathbf{L}_{\gamma_p(t)}))$.

(ii) $span(\tilde{\gamma}_p(\mathbb{R})) \subset \nu_0(\mathbf{L}_p)$.

(iii) For any fixed t *,* $\tilde{\gamma}_p(t)$ *is a parallel normal field to* $\mathbf{L}_p$ *and* $\mathbf{L}_{\gamma_p(t)} = (\mathbf{L}_p)_{\tilde{\gamma}_p(t)}$ *(i.e. the leaves of* $E_k^\perp$ *are parallel submanifolds of* $\mathbb{R}^N$ *).*

(iv) $\nu_0(\mathbf{L}_p)_p = \nu_0(M)_p \oplus df(E_k(p))$.

Proof . Let $v_p \in E_k^\perp(p)$ and let $v(t)$ be the parallel transport in M of v_p along $\gamma_p(t)$. Observe that $v(t) \in E_k^\perp(\gamma_p(t))$. Then, $\frac{d}{dt}v(t) = \nabla_{\gamma_p'(t)}v(t) + \alpha(\gamma_p'(t), v(t)) = 0$ (recall that $\alpha(E_k, E_k^\perp) = 0$ by Remark 2.2). This shows (i). Note that this implies that $\tilde{\gamma}_p(\mathbb{R}) \subset \nu(\mathbf{L}_p)_p$, since $\tilde{\gamma}_p(0) = 0$ and $\gamma_p'(t) \perp df(T_{\gamma_p(t)}(\mathbf{L}_{\gamma_p(t)})) = df(T_p(\mathbf{L}_p))$. Observe also that $df(X)|_{\mathbf{L}_p}$ is a parallel normal field to $\mathbf{L}_p$, for any $p \in M$, since $\|X\| = 1$ and $\alpha(X, E_k^\perp) = 0$. Then, using *(i)*, $\tilde{\gamma}_p'(t) \in \nu_0(\mathbf{L}_p)$. Since $\tilde{\gamma}(0) = 0$ we conclude that $span(\tilde{\gamma}_p(\mathbb{R})) \subset \nu_0(\mathbf{L}_p)$, which shows *(ii)*. It is now standard to show *(iii)*. Since the tangent space of $\mathbf{L}_p$ is invariant under the shape operators of M, we have that $\nu_0(M)_p \subset \nu_0(\mathbf{L}_p)_p$. Let us then decompose orthogonally

$$\nu_0(\mathbf{L}_p)_p = \nu_0(M)_p \oplus df(E_k(p)) \oplus V_p$$

Let $I = \mathbf{L}_p \cap \gamma_p(\mathbb{R})$. Since the foliations E_k and $E_k^\perp$ are transversal, I has countable many elements. Observe that for any $q \in I$ the set

$$H_q = \{\tau_c^\perp : V_p \to V_q\,;\ c \text{ is a curve in } \mathbf{L}_p \text{ from } p \text{ to } q\}$$

has also countable many elements, since $\nu_0(\mathbf{L}_p)$ is flat (observe that since $T(\mathbf{L}_p)$ is invariant under the shape operators of M, then $\tau^\perp_{c|V_p}$ can be also regarded as the $\nabla^\perp$-parallel transport in $\mathbf{L}_p$). Let now $\tau_q^\perp$ be the $\nabla^\perp$-parallel transport along γ_p from q to p. By making use of Sublemma 4.2 and Remark 4.3 we have that

$$\Phi^\perp_{p|V_p} = \{\tau_q^\perp \circ \tau\ :\ \tau \in H_q, q \in I\}$$

has countable many elements. Then $V_p \subset \nu_0(M)_p$ (recall that M is simple connected). □

Remark 4.8 Drop, in the lemma above, the conditions that M is complete and that the number of curvature normals is constant. Let U be an open subset of M and let η be a C^∞ curvature normal of multiplicity 1 defined in U and such that its associated eigendistribution E is autoparallel and $E^\perp$ is integrable. Then the same conclusions of the Lemma 4.7 are true locally in U, provided $rank_f^{loc}(M) = rank_f(M)$. The proofs are the same. We will make use of this observation for the proof of Theorem 1.1

It is standard to show that the projection to $\nu_0(M)_q$ of $\alpha(X(q), X(q)) = \gamma_q''(0)$ coincides with $\eta_k(q)$, for all $q \in M$. But, by Lemma 4.7, we have that this second derivative belongs to $\nu_0(M)$, since it belongs to $\nu_0(\mathbf{L_p})$ and it is perpendicular $\tilde{\gamma}_q'(0) = df(X(q))$. The same is true locally by Remark 4.8. So, we have the following corollary.

Corollary 4.9 *We are under the assumptions and notation of Lemma 4.7 or Remark 4.8. Then*

$$\eta(\gamma_p(t)) = \tilde{\gamma}_p''(t)$$

(where $\eta := \eta_k$ in the case of Lemma 4.7)

Proof. Observe that $p \mapsto \tilde{\gamma}_p'(0)$ is a parallel normal field to $\mathbf{L}_p$ and so, $p \mapsto \tilde{\gamma}_p''(0)$ is also a parallel normal field to $\mathbf{L}_p$. But $\tilde{\gamma}_p''(0) \perp \tilde{\gamma}_p'(t)$. So, by part *(iv)* of Lemma 4.7, we obtain the corollary for $t = 0$. For $t \neq 0$, just observe that $\tilde{\gamma}_{\tilde{\gamma}_p(t)}(s) = \tilde{\gamma}_p(t+s)$

Remark 4.10 Actually, the above corollary is equivalent to part *(iv)* of Lemma 4.7. Namely, a direct proof of the above corollary can be done using Sublemma 4.2 and the lemma in [O1, appendix]. From this it is not hard to deduce part *(iv)* of Lemma 4.7.

The following lemma relates the curvature normals of the isometric immersion $f : M \to \mathbb{R}^N$ to the curvature normals of the leaves of ${E_k}^\perp$.

Lemma 4.11 *We are under the assumptions of Lemma 4.1 (the completeness of M may be replaced by the assumption that E_k is autoparallel and $E_k^\perp$ is integrable). Then*

(i) The eigenspaces of the simultaneous diagonalization of the shape operators of $\nu_0(\mathbf{L})_p$ coincides with the eigenspaces $E_1(p), \ldots, E_{k-1}(p), E_{k+1}(p), \ldots, E_g(p)$.

(ii) Let $\tilde{\eta}_1(p), \ldots, \tilde{\eta}_{k-1}(p), \tilde{\eta}_{k+1}(p), \ldots, \tilde{\eta}_g(p)$ be the curvature normals at p of the leaf $\mathbf{L}_p$ of $E_k^\perp$ (regarded as a submanifold of the ambient space) associated to the decomposition $T_p.\mathbf{L}_p = E_1(p) \oplus \cdots \oplus E_{k-1}(p) \oplus E_{k+1}(p) \oplus \cdots \oplus E_g(p)$. Then

$$\eta_i(\gamma_p(t)) = \frac{\tilde{\eta}_i(p)}{1 - \langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle} - \langle \frac{\tilde{\eta}_i(p)}{1 - \langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle}, \tilde{\gamma}_p'(t) \rangle \, \tilde{\gamma}_p'(t)$$

Proof. Part *(i)*: Let $\tilde{A}$ denote the shape operator of the leaves of $E_k^\perp$, regarded as submanifolds of the ambient space, and let X be the unit tangent field to M which generates E_k. It suffices to show, by part *(iv)* of Lemma 4.7, that $\tilde{A}_X$ leaves invariant E_i and that it is a multiple of the identity ($i \neq k$). Let X_i, Y_j be tangent field to M which lie in E_i and E_j respectively. Let ξ be a (locally defined) parallel normal section of M which distinguish all of the different eigenvalues $\lambda_1 = \langle \eta_1, \ \rangle, \cdots, \lambda_g = \langle \eta_g, \ \rangle$ $(i, j \neq k)$. The Codazzi identity $\langle (\nabla_{Y_j} A_\xi)(X), X_i \rangle = \langle (\nabla_{X_i} A_\xi)(X), Y_j \rangle$ implies, by a direct computation, that

$$(*) \qquad \langle \xi, \eta_k - \eta_i \rangle \langle \nabla_{Y_j} X, X_i \rangle = \langle \xi, \eta_k - \eta_j \rangle \langle \nabla_{X_i} X, Y_j \rangle$$

But $\langle \nabla_{X_i} X, Y_j \rangle = -\langle \tilde{A}_X X_i, Y_j \rangle = \langle \nabla_{Y_j} X, X_i \rangle$. Then $(*)$ implies, if $i \neq j$, that $\langle \tilde{A}_X X_i, Y_j \rangle = 0$. Then E_i is invariant under the shape operators of $\mathbf{L}_p$. Let now $i = j$ and assume $X_i \perp Y_i$. A direct computation shows that $\langle (\nabla_X A_\xi)(X_i), Y_i \rangle = 0$. So, by the Codazzi identity we have that $0 = \langle (\nabla_{X_i} A_\xi)(X), Y_i \rangle = \langle \xi, \eta_k - \eta_i \rangle \langle \nabla_{X_i} X, Y_i \rangle$. This implies that $\langle \tilde{A}_X(X_i), Y_i \rangle = 0$. Then $\tilde{A}_{X|E_i}$ must be a multiple of the identity. This implies part (i).
Part (ii): from the well known formula relating shape operators of parallel manifolds (see [HOT]), one has that the curvature normals of $\mathbf{L}_{F_t(p)}$ are $\frac{\tilde{\eta}_i(p)}{1-\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle}$, $i \neq k$. If π^t denotes the orthogonal projection to $(\tilde{\gamma}_p'(t))^\perp$ then it is not hard to see that $\eta_i = \pi^t(\frac{\tilde{\eta}_i(p)}{1-\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle})$, which implies part (ii). $\square$

Remark 4.12 Using the Codazzi identity, if $\|X_i\| = 1$, $\langle (\nabla_X A_\xi)(X_i), X_i \rangle = \langle (\nabla_{X_i} A_\xi)(X), X_i \rangle$ one has that

$$X(\langle \xi, \eta_i \rangle) = \langle \xi, \eta_i - \eta_k \rangle \langle \tilde{A}_X(X_i), X_i \rangle$$

if $\nabla^\perp \xi = 0$.

Now, we are ready to prove Theorem 1.1 under the assumption that the number of curvature normals is constant on M

Proof of Theorem 1.1 (in the case that the number of curvature normals is constant on M). Assume first, that the number of the curvature normals is constant on M. So, by part (ii) of Lemma 4.11 we have that the curvature normals satisfies for each $i = 1, 2, \ldots, k-1, k+1, \ldots, g$ the following equation:

$$\langle \eta_i(\gamma(t)), \eta_i(\gamma(t)) \rangle = \langle \frac{\tilde{\eta}_i(p)}{1-\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle}, \frac{\tilde{\eta}_i(p)}{1-\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle} \rangle - \langle \frac{\tilde{\eta}_i(p)}{1-\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle}, \tilde{\gamma}_p'(t) \rangle^2$$

Let $c_i := \langle \eta_i, \eta_i \rangle$ and $\tilde{c}_i := \langle \tilde{\eta}_i(p), \tilde{\eta}_i(p) \rangle$ (observe that $0 \leq c_i \leq \tilde{c}_i$, since $\eta_i(p)$ is the orthogonal projection of $\tilde{\eta}_i(p)$ to $\nu_0(M)_p$). Then, the function $f_i(t) := 1 - \langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t) \rangle$ satisfies the following differential equation

$$(A) \qquad c_i f_i(t)^2 = \tilde{c}_i - f_i'(t)^2$$

with initial condition $f_i(0) = 1$ By taking derivatives in the above equation one has that $2f_i'(c_i f_i + f'') = 0$. From this it is standard to prove that either f_i satisfies, for all t,

$$(B) \qquad f_i'(t) = 0$$

or, for all t, f_i satisfies

$$(C) \qquad c_i f_i(t) + f_i''(t) = 0$$

So, $f_i(t) \equiv 1$ or $f_i(t) = \frac{sin(\sqrt{c_i}(t+t_0))}{sin(\sqrt{c_i}t_0)}$, where t_0 satisfies $cotg^2(\sqrt{c_i}t_0) = \frac{\tilde{c}_i - c_i}{c_i}$ (evaluating equation *(A)* at $t = 0$. This last case is impossible for our f_i because this implies that we can not pass to a parallel leaf when $\sqrt{c_i}(t+t_0)$ is a root of $sin(x) = 0$ (recall that M is complete). So, we have that $\langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t)\rangle = 0$ which implies, by taking second derivatives, $\langle \eta_k, \eta_i \rangle = 0$ on M, for $i \neq k$, since $\eta_k(\gamma_p(t)) = \tilde{\gamma}_p''(t)$ due to Corollary 4.9

We will now prove that $f : M \to \mathbb{R}^N$ splits. Note that E_k is invariant by the shape operator of M as it follows from Remark 2.2. Since M is simple connected it will suffice to show that $E_k^\perp$ is also autoparallel (recall that if the orthogonal complement of an autoparallel distribution is autoparallel then both distributions must be parallel). In fact, since M is simple connected it must split and then we apply Moore's Lemma [M] to split the immersion. Let us then show that $E_k^\perp$ is an autoparallel distribution of M. Observe that $\tilde{A}_X$ coincides with the shape operator of the leaves of $E_k^\perp$ regarded as hypersurfaces of M, where $\tilde{A}$ denotes the shape operator of the leaves of $E_k^\perp$, regarded as submanifolds of the ambient space. We claim that $\tilde{A}_X \equiv 0$. In fact, let $q \in M$ be fixed and let ξ^q be a parallel normal section to M such that $\xi^q(q) = \eta_k(q)$. Then the first side of the equation in Remark 4.12 vanishes at q, because the function $\langle \xi^q, \eta_k \rangle$ has a maximum at q (using Cauchy-Schwartz inequality, since ξ^q and η_k have both constant length). The second hand of the equality of Remark 4.12 implies, since $\langle \xi^q, \eta_i(q)\rangle = 0$, that $\langle \tilde{A}_X(X_i), X_i\rangle_q = 0$. Since q is arbitrary we obtain that $\tilde{A}_X \equiv 0$. So, we have shown that if some curvature normal is not $\nabla^\perp$-parallel we can split globally the immersion $f : M \to \mathbb{R}^N$. This complete proof of Theorem 1.1 under the assumption that the number of curvature normals is constant on M. $\square$

Remark 4.13 We are under the assumptions and notation of Lemma 4.11. Then $\langle \eta_k(\gamma_p(t)), \eta_i(\gamma_p(t))\rangle$ is constant (of course, if M is complete this turns out to be zero as it follows from the proof above). In fact, as in the proof of Theorem 1.1, $f_i(t) = 1 - \langle \tilde{\gamma}_p(t), \tilde{\eta}_i\rangle$ satisfies either $f_i' = 0$ or $-f_i'' = c_i f_i$, where $c_i = \langle \eta_i, \eta_i\rangle \geq 0$. On the other hand

$$-f_i''(t) = \langle \tilde{\gamma}_p''(t), \tilde{\eta}_i(p)\rangle = \langle \eta_k(\gamma_p(t)), \tilde{\eta}_i(p)\rangle$$

by Corollary 4.9. Then

$$c_i = \frac{-f_i''(t)}{f_i(t)} = \langle \eta_k(\gamma_p(t)), \frac{\tilde{\eta}_i(p)}{f_i(t)}\rangle$$

$$= \langle \eta_k(\gamma_p(t)), \frac{\tilde{\eta}_i(p)}{f_i(t)} - \langle \frac{\tilde{\eta}_i(p)}{f_i(t)}, \gamma_p'(t)\rangle\, \gamma_p'(t)\rangle = \langle \eta_k(\gamma_p(t)), \eta_i(\gamma_p(t))\rangle$$

by Lemma 4.11 (in the case f_i satisfies the second differential equation). If $f_i' = 0$, then the same computations shows that $\langle \eta_k(\gamma_p(t)), \eta_i(\gamma_p(t))\rangle = 0$ and so it is also constant.

Proof of Theorem 1.1 (in the case that $rank_f^{loc}(M) = rank_f(M)$). Let Ω be the open an dense subset of M where the number of curvature normals is locally constant (in this way any point of Ω has a neighbourhood where the curvatures normals define C^∞ normal fields to M. Let $l_1 > \cdots > l_r$ be the different lengths of curvature normals (recall that $L = \{l_1, \cdots, l_r\}$ is the (constant) spectrum of the tensor B^0; see §1;2). Let us consider, for $l \in L$, the condition

($\mathbf{C}_l$) *Let η be a C^∞ curvature normal of length l defined in a neighbourhood of $p \in \Omega \subset M$ where its associated multiplicity is constant. Then either η is parallel in some neighbourhood of p or $\eta(p)$ is perpendicular to any other curvature normal at p.*

We will show that for any $l \in L$ the condition ($\mathbf{C}_l$) is always satisfied. If fact, if not, let $l \in L$ be the biggest length such that condition ($\mathbf{C}_l$) is not true. So, there exists $p \in \Omega$ and a C^∞ curvature normal η with $\|\eta\| = l$ defined around p and such that it is neither parallel in any neighbourhood of p nor perpendicular to all other curvature normal at p. We may assume, eventually by choosing another point near p, that $(\nabla^\perp)_p \eta \neq 0$ and that η is not perpendicular some curvature normal at p. By hypothesis, any curvature normal $\tilde{\eta}$ which is longer than η is either parallel around p or perpendicular to $\eta(p)$ (and to any other curvature normal at p). So, in an appropriate neighbourhood of p we are in the hypothesis of Lemma 4.6. Then the eigendistribution E associated to η, near p, is one dimensional, autoparallel and $E^\perp$ is integrable. Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic with $\gamma(0) = p$, $\gamma'(0) \in E(p)$. Then γ is, near 0, an integral manifold of E. We need the following

Auxiliary Lemma. *The geodesic γ, near any fixed $t \in \mathbb{R}$, is an integral manifold of an autoparallel (one dimensional) eigendistribution E_t (with associated curvature normal η^t), defined in a neighbourhood V_t of $\gamma(t)$ and such that $E_t^\perp$ is integrable.*

Proof. First of all observe that if one shows this lemma then, if $\gamma_(t_0)$ belongs to $V_{t_1} \cap V_{t_2}$, $\eta^{t_1} = \eta^{t_2}$ and $E_{t_1} = E_{t_2}$ in a neighbourhood of $\gamma(t_0)$ (see Remark 2.3 *(ii)*).

Let $b \in \mathbb{R}$ be the infimum of the set which consists of those $t > 0$ such that the conclusion of this lemma is false (observe that $b > 0$ by what remarked before this lemma). Assume $b < +\infty$. Since near any $t \in [0, b)$ the lemma is true, let $\mathbf{L}_{\gamma(t)}$ be the integral manifold of $E_t^\perp$ through $\gamma(t)$. Then, as in the proof of Lemma 4.7 part *(iii)*, we have that, via the immersion f, $\mathbf{L}_{\gamma(t)}$ is a parallel manifold to $\mathbf{L}_{\gamma(t')}$, for all $t, t' \in [0, b)$. Namely $(\mathbf{L}_{\gamma(t)})_{f(\gamma(t')) - f(\gamma(t))} = \mathbf{L}_{\gamma(t')}$. Let $\bar{\eta}_1^p, \cdots, \bar{\eta}_g^p$ be the other distinct curvature normals at p. As in Remark 4.13 the curvature normals $\bar{\eta}_i^p(t)$ along $\gamma(t)$ have the property that $\langle \bar{\eta}_i(t), \eta^t(\gamma(t)) \rangle$ is constant for all $i = 1, \cdots, g$, $0 \leq t < b$. The proof is the same as that of Remark 4.13. One should notice that since the curvature normals are not C^∞ globally defined we cannot use the notation $\bar{\eta}_i(\gamma(t))$ (it also may happen that $\gamma(t)$ goes outside Ω). This implies, since the curvature normals have constant length, that $\bar{\eta}_i(t)$ cannot collapse to the multiplicity 1 curvature normal $\eta^t(\gamma(t))$ as t tends to b. Then $\eta^t(\gamma(t))$ converges, as t tends to b, to a multiplicity 1 curvature normal at $\gamma(b)$ (using the property **(*)** of section 2). By Remark 2.3 part *(ii)*, we obtain that this curvature normal at $\gamma(b)$ extends to a C^∞ curvature normal η^b of constant multiplicity 1 with associated eigendistribution E_b defined in a neighbourhood $\tilde{V}_b$ of $\gamma(b)$. The distribution E_b is autoparallel in $\tilde{V}_b \cap \Omega$, by the same reason that $E_0 = E$ is so, since $\|\eta^t(\gamma(t))\| = l$ (see the paragraph before the lemma). Since $\tilde{V}_b \cap \Omega$ is dense in $\tilde{V}_b$ we obtain the autoparallelity of E_b in $\tilde{V}_b$. We will prove, perhaps by making $\tilde{V}_b$ smaller, that $E_b^{\perp}$ is integrable. We may assume that there exists a unit vector field X in $\tilde{V}_b$ such that it spans E_b. Let F_s be the flow of X. Then, as in the proof of Lemma 4.1, F_s preserves $E_b^{\perp}$. There must exist $\varepsilon > 0$ and an open neighbourhood $U \subset \tilde{V}_b$ of $\gamma(b)$ such that F_s is defined in U for all $s \in (-\varepsilon, \varepsilon)$. Let $t_0 \in (b-\varepsilon, b)$. We may assume that the integral manifold $\mathbf{L}_{\gamma(t_0)}$ of E_{t_0} through $\gamma(t_0)$ is contained in U. Then $V_b = \{F_s(\mathbf{L}_{\gamma(t_0)}) : s \in (-\varepsilon, \varepsilon)\}$ is an open subset of $\tilde{V}_b$ which contains $\gamma(b)$. Moreover, E_b is integrable on V_b with leaves $F_s(\mathbf{L}_{\gamma(t_0)})$. Then, if we set $V_t = V_b$, $E_t = E_b$ for $t > b$, t close to b, we obtain a contradiction. Then $b = +\infty$ and the lemma is true for $t > 0$. For $t < 0$ just change the orientation of γ. $\square$

We continue with the proof of our theorem. We use the notation of the proof of the Auxiliary Lemma. We have, as in the proof of this theorem, in the case of constant number of curvature normal, that $f_i(t) := 1 - \langle \tilde{\eta}_i(p), \tilde{\gamma}_p(t)\rangle$ satisfies either the differential equation $f_i'(t) \equiv 0$ or $f_i(t)c_i + f_i''(t) \equiv 0$, where $\tilde{\eta}_i$ is a curvature normal of the leaf $\mathbf{L}_p$. If it satisfies the second differential equation, in the same way as for the case of constant number of curvatures normals, we obtain a contradiction. So, $f_i'(t) \equiv 0$ and then, by Remark 4.13 and Corollary 4.9, $\langle \tilde{\eta}_i(t), \eta^t(\gamma(t))\rangle \equiv 0$. We obtain, making $t = 0$, that $\eta(p)$ is perpendicular to any other curvature normal at p. A contradiction. So, the condition $\mathbf{C}_l$ is satisfied for all $l \in \mathbf{L}$. This implies that the scalar product of any two C^∞ curvature normals, defined in an open subset of Ω is constant. Then, by Lemma 2.7, the number of curvature normals is constant on M. But this is the case we have already solved. $\square$

5 Local counterexamples

In this section we will explicitly construct examples which show that Theorem 1.1 is false if M is not assumed to be complete. We will find a 1-parameter family f_d of isometric immersions from the 2 dimensional sphere $\mathbb{S}^2$ without two points into $\mathbb{R}^4$. These immersions have flat normal bundle and curvature normals of constant length but they are not isoparametric. Similar constructions can be done by starting with the n dimensional sphere without two points.

Let us begin with some general facts. Let $f : M^n \to \mathbb{R}^N$ be an immersed submanifold with flat normal bundle and let $c : (a,b) \to \nu(M)_p$ be an arc-length parameterized curve with $c(0) = 0$ ($p \in M$ is fixed). Let, for any $t \in (a,b)$, ξ_t be the unique parallel normal field to M with $\xi_t(p) = c(t)$ Assume that $\xi_t(q)$ does not belong to the focal hyperplanes at q, for all $t \in (a,b)$ and for $q \in M$ (i.e. 1 in not an eigenvalue of the shape operator A_{ξ_t} for all $t \in (a,b)$. In this way, for any $t \in (a,b)$, it is defined the parallel manifold $f_t : M \to \mathbb{R}^N$, where $f_t(q) = f(q) + \xi_t(q)$. This parallel manifold is often denoted by M_{ξ_t}, which is nothing else than M, but with the metric induced by f_t (locally the parallel manifold is identified with $f_t(M)$; see [HOT]). The immersion f can be extended to an immersion $\tilde{f} : \tilde{M} \to \mathbb{R}^N$ where $\tilde{M} = (a, b) \times M$ and

$$\tilde{f}(t, q) = f_t(q)$$

It is standard to show that $\tilde{M}$ (with the metric induced by $\tilde{f}$) is also a submanifold with flat normal bundle of $\mathbb{R}^N$. Let us summarize, in next lemma, the main properties of the submanifold $\tilde{M}$. The proof use similar arguments to those used in Lemma 4.7.

Lemma 5.1 *Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be the integrable distributions on $\tilde{M}$ whose integral manifolds are $(a, b) \times \{q_0\}$ and $\{t_0\} \times M$ respectively. Then*

1) $\mathcal{D}^1 \perp \mathcal{D}^2$. Moreover, $\mathcal{D}^1$ and $\mathcal{D}^2$ are both invariant under the shape operator $\tilde{A}$ of $\tilde{M}$ (or equivalent, $\tilde{\alpha}(\mathcal{D}^1, \mathcal{D}^2) = 0$).

2) $\gamma_q(t) = (t, \{q\})$ is a (unit speed) geodesic of $\tilde{M}$ (with the metric induced by $\tilde{f}$) and so $\mathcal{D}^1$ is an autoparallel distribution.

3) The integral manifolds of $\mathcal{D}^2$ are parallel manifolds of $\mathbb{R}^N$ (via $\tilde{f}$). Namely, $\{t_0\} \times M = (\{0\} \times M)_{\xi_{t_0}}$.

4) $\tilde{M}$ has flat normal bundle. Moreover, let $(t_0, q_0) \in \tilde{M}$ and let $E_1, \cdots, E_g$ be the common eigenspaces at (t_0, q_0) determined by the commuting family of shape operators of $\tilde{f} : \{t_0\} \times M \to \mathbb{R}^N$ with associated curvature normals $\eta_1, \cdots, \eta_q$. Then the common eigenspaces for the shape operator $\tilde{A}$ of $\tilde{M}$ at (t_0, q_0) are given by $\mathcal{D}^1_{(t_0,q_0)}, E_1, \cdots, E_g$ with associated curvature normals $\tilde{\gamma}''_{q_0}(t_0), \pi^{t_0}_{q_0}(\eta_1), \cdots, \pi^{t_0}_{q_0}(\eta_g)$, where $\pi^{t_0}_{q_0}$ is the orthogonal projection in $\mathbb{R}^N$ to the orthogonal complement of $\mathcal{D}^1_{(t_0,q_0)}$.

Remark 5.2 In the lemma above the manifold $\tilde{f} : \{t_0\} \times M \to \mathbb{R}^N$ is a parallel manifold to $\tilde{f} : \{0\} \times M \to \mathbb{R}^N$. Namely, $\{t_0\} \times M = (\{0\} \times M)_{\xi_{t_0}}$. So, if $\tilde{\eta}_1, \cdots, \tilde{\eta}_g$ are the curvature normals of $\{0\} \times M$ at $(0, q)$, then the curvature normals of $\{t_0\} \times M$ at (t, q) are given by

$$\eta^t_1 = \frac{\eta_1}{1 - \langle \eta_1, \xi_t(q) \rangle}, \cdots, \eta^t_g = \frac{\eta_1}{1 - \langle \eta_g, \xi_t(q) \rangle}$$

where $\xi_t(q)$ is the normal parallel transport of $c(t)$ from p to q.

Let us analyze the particular case when $M = S^1 \subset \mathbb{R}^{N+2} = \mathbb{R}^2 \times \mathbb{R}^N$ is the extrinsic circle of radius 1 which is contained $\mathbb{R}^2 \times \{\mathbf{0}\}$. Let r_θ be the (positive) rotation of angle θ in $\mathbb{R}^2$ and extend it, let us say to $\bar{r}_\theta$, in a natural way to $\mathbb{R}^{N+2}$. Namely, $\bar{r}_\theta((x, y)) = (r_\theta(x), y)$, where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^N$. Let $c : (a, b) \to \nu_{e_1} S^1 = e_2^\perp$ be an arc-length parameterized curve with $c(0) = 0$ (where $e_1 = (1, 0, \mathbf{0})$, $e_2 = (0, 1, \mathbf{0}) \in \mathbb{R}^{N+2}$). Let us consider, as in Lemma 5.1, the immersion (t small in order that $\|c(t)\|$ be less than 1, the focal distance of S^1) $\tilde{f} : (a, b) \times S^1 \to \mathbb{R}^{N+2}$ defined by

$$\tilde{f}(t, \bar{r}_\theta(e_1)) = \bar{r}_\theta(e_1) + \xi_t(\bar{r}_\theta(e_1)) = \bar{r}_\theta(e_1) + \bar{r}_\theta(c(t))$$

since $\bar{r}_\theta$ gives the parallel transport in the normal space to S^1.
Let us write $c(t) = (x(t), 0, \bar{c}(t))$ where $x : (a, b) \to \mathbb{R}$ and $\bar{c} : (a, b) \to \mathbb{R}^N$.
By Lemma 5.1 the immersed submanifold $\tilde{f} : (a, b) \times S^1 \to \mathbb{R}^{N+2}$ has flat normal bundle, for t small. Moreover, since $-\bar{r}_\theta(e_1)$ is the curvature normal of S^1 at $\bar{r}_\theta(e_1)$, the submanifold $\tilde{f} : (a, b) \times S^1 \to \mathbb{R}^{N+2}$ has (at most) two curvature normals at $(t, \bar{r}_\theta(e_1))$ given by (see Remark 5.2)

$$\eta_1((t, \bar{r}_\theta(e_1))) = \bar{r}_\theta(x''(t), 0, \bar{c}''(t))$$

and

$$\begin{aligned}\eta_2((t, \bar{r}_\theta(e_1))) &= \bar{r}_\theta(-\frac{e_1}{1 + x(t)} + \frac{x'(t)}{1 + x(t)}(x'(t), 0, \bar{c}'(t))) \\ &= \bar{r}_\theta((\frac{x'(t)^2 - 1}{1 + x(t)}, 0, \frac{x'(t)\bar{c}'(t)}{1 + x(t)}))\end{aligned}$$

By a direct computation, if $\bar{x}(t) = 1 + x(t)$, one has that

$$\|\eta_1((t, \bar{r}_\theta(e_1)))\|^2 = \bar{x}''(t)^2 + \|\bar{c}''\|^2$$

and

$$\|\eta_2((t, \bar{r}_\theta(e_1)))\|^2 = \frac{1 - \bar{x}'(t)^2}{\bar{x}(t)^2}$$

(using that $\|\bar{c}'(t)\|^2 = 1 - x'(t)^2$, since $||c'(t)||^2 = 1$).

So,

Lemma 5.3 *$\tilde{f} : (a, b) \times S^1 \to \mathbb{R}^{N+2}$ has curvature normals of constant length if and only if there exists constants $d, \tilde{d} \geq 0$ such that:*

i) $\bar{x}''(t)^2 + \|\bar{c}''(t)\|^2 = d$.

ii) $\frac{1 - \bar{x}'(t)^2}{\bar{x}(t)^2} = \tilde{d}$,

where $c(t) = (\bar{x}(t) - 1, 0, \bar{c}(t))$

We will construct explicitly the counterexamples for $\tilde{d} = 1$ and $N = 2$. In this case $\bar{c}(t)$ is a plane curve contained in $\{(0, 0)\} \times \mathbb{R}^2$. Observe that $\bar{x}(t) = \cos(t)$ verifies condition (ii) of the previous lemma (observe that $\bar{x}(0) = 1$ and $\bar{c}(0) = 0$, since $c(0) = 0$).
Assume $\bar{c}(t) = \tilde{c}(\sin(t))$ where $\tilde{c}$ is parameterized by arc length. Condition (i) of the lemma above translate into the following equation for the curvature of the plane curve $\tilde{c}(s)$, $-1 < s < 1$:

$$(I) \qquad \kappa(s) = \frac{\sqrt{d-1}}{1-s^2}$$

So, let $\tilde{c}$ be the unique arc-length parameterized plane curve (contained in $\{(0,0)\} \times \mathbb{R}^2$) whose curvature satisfies the equation *(I)* ($d > 1$) with initial conditions: $\tilde{c}(0) = 0$ and $\tilde{c}'(0) = e_3 = (0,0,1,0)$ (From *(I)* one has that $\tilde{c}(t) = \tilde{c}(-t)$). Then the curve $c(t) = (\cos(t) - 1, 0, \tilde{c}(\sin(t))$ defines the immersed submanifold $\tilde{f}_d : (-\pi, \pi) \times S^1 \to \mathbb{R}^4$ which has curvature normals of constant length $\sqrt{d} > 1$ and 1. A direct computation shows that $\langle \eta_1, \eta_2 \rangle = 1$. So, using the Gauss equation, one has that $((-\pi,\pi) \times S^1, \langle \, , \, \rangle_d)$ has constant curvature 1, where $\langle \, , \, \rangle_d$ is the metric induced by $\tilde{f}_d$ (in particular the immersion $\tilde{f}_d$ cannot be a product of immersions). Observe that $((-\pi,\pi) \times S^1, \langle \, , \, \rangle_d)$ is (intrinsically) the unit sphere without two antipodal points, since $c(t) = c(-t)$. Notice also that the curvature normals η_1 and η_2 cannot be both parallel. Otherwise $\tilde{f}_d : (-\pi,\pi) \times S^1 \to \mathbb{R}^4$ would be isoparametric and so a product of extrinsic circles, since it has two different curvature normals.

Notice that, for d close to 1, $\tilde{f}_d$ must be an embedding. As d increases, $\tilde{f}_d$ becomes an isometric immersion with a lot of auto-intersections and a chaotic behavior (as it follows from the study of the graphs of the curves $\tilde{c}(s)$).

Remark 5.4

i) Similar constructions can be carried out for $N > 2$. In this case the curve $\tilde{c}$ is not contained any more in a plane. The condition that the curvature normals of the immersion $\tilde{f}_d$ have constant length is equivalent to the fact that the (first) curvature κ_1 satisfies equation *(I)*. The higher order curvatures $\kappa_2, \cdots \kappa_{N-1}$ can be arbitrary positive C^∞ functions (which are the coefficients of an adapted Frenet frame). This shows that there are non analytic examples in higher codimension.

ii) Similar construction can be done by beginning with an n-dimensional sphere S^n instead of a circle. In that case one produces non isoparametric isometric immersions from the S^n without two antipodal points (with flat normal bundle and two curvature normals of constant length)

Question. Any submanifold with flat normal bundle and curvature normals of constant length can be produced as in Lemma 5.1 (this is by what done in section 4). The question is whether one can construct (irreducible and full) examples by beginning with an arbitrary submanifold different from the sphere. Equivalently: *does there exist a non isoparametric irreducible submanifold of euclidean space, with flat normal bundle, and at least three curvature normals of constant length?*

Remark 5.5 Observe that in the examples the lengths of the (two) curvature normals are not equal. If the curvature normals are all of the same (constant) length, in the case of flat normal bundle, then, due to result of Nölker [N], the submanifold must be a sphere, provided it is irreducible. Actually, the Theorem of Nölker follows from Lemma 4.6. In fact, $E^{\perp}$ must be not only integrable but also autoparallel (this is a consequences, using Codazzi identity, of the fact that it is integrable and the sum of totally geodesic eigendistributions, see [O2, pp. 624]). So, by Moore's Lemma [M] the submanifold splits. So, there is only one eigendistribution and the submanifold is totally umbilical.

Further comments. For the third fundamental form of a submanifold M of euclidean space one has the following identity (see e.g. [N])

$$B = A_H - Ric$$

where A is the shape operator, H is the mean curvature vector and Ric is the intrinsic Ricci endomorphism.
Then, if M is a minimal submanifold, $B = -Ric$. In this case, B has constant eigenvalues if and only if Ric has constant eigenvalues. Assume that M has flat normal bundle and that its Ricci endomorphism has constant eigenvalues (this last condition is always fulfilled if M is intrinsically homogeneous). Then from Corollary 1.4 it follows that M is isoparametric. Then M is totally geodesic since it is minimal (observe that an isoparametric submanifold which does not split a compact factor must be totally geodesic [PT]). This extends the result of Marcos Dajczer that there are no minimal isometric immersion with flat normal bundle from hyperbolic space into euclidean space. (see Theorem 4.9 in [D]). The question is whether this is also true without assuming that M has flat normal bundle. In particular: *does a homogeneous non-flat riemannian manifold M admit a minimal isometric immersion into euclidean space*. If M is the hyperbolic plane then the answer is negative by [Di] (Observe that a Ricci flat minimal euclidean submanifold must be totally geodesic, since $B = 0$ if and only if $\alpha = 0$).

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