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Generalized Local Equilibrium in the Cascaded Lattice Boltzmann Method

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Abstract

By realizing the insufficient degree of Galilean invariance of the traditional multiple–relaxation–time (MRT) collision operators, Geier et al. [Phys. Rev. E 73, 066705 (2006)] proposed to relax differently the moments shifted by the macroscopic velocity, leading to the so-called cascaded lattice Boltzmann method (LBM). This paper points out that (A) the cascaded LBM essentially consists in adopting a generalized local equilibrium in the frame at rest; (B) this new equilibrium does not affect the consistency of LBM; finally (C), if the raw moments are relaxed in the frame at rest as usual and the number of relaxation frequencies is reduced, the proposed derivation leads to the two–relaxation–time (TRT) collisional operator with proper polynomial equilibrium.

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I. INTRODUCTION

The lattice Boltzmann method (LBM) is considered a viable alternative for solving the hydrodynamic Navier–Stokes equations [1–3]. The Lax equivalence theorem remembers us that (a) consistency and (b) stability are two essential conditions for ensuring the convergence of the numerical solution to the well–posed initial value problem [4]. Proving (a) the consistency of LBM, with regards to the Navier–Stokes equations, can be done, for example, by the Chapman–Enskog expansion [5, 6] or by the Hilbert expansion with proper scaling [7, 8]. Unfortunately a mathematical tool for analyzing in general (b) the stability of a system of non–linear partial differential equations is currently missing. A popular approach lays on linearizing the system of equations around an arbitrary configuration, applying a Fourier transform in order to get rid of the spatial gradients in case of periodic boundaries and finally discussing the obtained ordinary differential equations by the von Neumann analysis [4, 9, 10]. However, in general, many heuristic issues are proposed for guiding the design of stable LBM schemes, including how to discretize the velocity space [11–15] and how to truncate the polynomial expansion of the local equilibrium [14].

Certainly the collision step of the algorithm has been proved to play an essential role. In particular, the *multiple–relaxation–time* (MRT) *collisional operator*, which was first heuristically proposed in order to enhance collisions [16], then systematically developed [9, 17], and its variants, such as the two–relaxation–time (TRT) operator [18], allow one to enhance the stability, by properly tuning the numerical bulk viscosity, which is a free parameter in a scheme aiming to recover the incompressible limit of Navier–Stokes equations.

Recently a new result has been added to the previous picture. By realizing the insufficient degree of Galilean invariance of the traditional MRT collision operators, Geier et al. [19] proposed to relax differently the *central moments*, i.e. the moments shifted by the macroscopic velocity, in a *moving frame* (instead of the traditional practice of relaxing the raw moments in the frame at rest), leading to the so-called cascaded LBM.

This paper aims to provide a simple mathematical interpretation, pointing out that (A) the cascaded LBM essentially consists in adopting a *generalized local equilibrium* in the frame at rest, which is a function of both conserved and non–conserved hydrodynamic moments. Moreover (B) the asymptotic analysis proves that the method consistently recovers the correct system of macroscopic equations. Finally (C), despite the different formalism, if the raw moments are relaxed in the frame at rest as usual and the number of relaxation frequencies is reduced, the proposed derivation leads to the two–relaxation–time (TRT) collisional operator with proper polynomial equilibrium.

This paper is organized as follows. In Section II, some preliminaries are introduced. In Section III, two derivations are reported, based on relaxing the raw moments in the frame at rest as usual (result C) and on relaxing the central moments in the moving frame, leading to the cascaded LBM and the generalized local equilibrium (result A). In Section IV, it is proved that the generalized local equilibrium does not affect the consistency of the LBM (result B). Finally some conclusions are reported.

II. PRELIMINARIES

A. Continuous velocity space

Let us introduce the local equilibrium distribution function φ_{eq} in the continuous two–dimensional velocity space $(\xi_x, \xi_y) \in \mathbb{R}^2$, namely

$$\varphi_{eq} = \frac{3\,\bar{\rho}}{2\,\pi}\,\exp\left[-\frac{3\,(\xi_i - \bar{u}_i)^2}{2}\right], \tag{1}$$

where $\bar{\rho} = \langle\langle\varphi\rangle\rangle$, $\bar{\rho}\bar{u}_i = \langle\langle\xi_i\,\varphi\rangle\rangle$ $(i = x, y)$, φ is the generic distribution function and

$$\langle\langle\cdot\rangle\rangle = \int_{-\infty}^{+\infty} \cdot\, d\xi_x d\xi_y. \tag{2}$$

It is possible to prove that the continuous local equilibrium given by Eq. (1) minimizes an entropy function $H(\varphi)$, under the constraints of mass and momentum conservation [14].

Let us introduce the generic continuous *raw* equilibrium moment

$$\gamma^{eq}_{xx\cdots x\,yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}}, \overbrace{yy\cdots y}^{m\text{ times}}) = \langle\langle\xi_x^n\xi_y^m\varphi_{eq}\rangle\rangle, \tag{3}$$

and the corresponding continuous *central* equilibrium moment

$$\hat{\gamma}^{eq}_{xx\cdots x\,yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}}, \overbrace{yy\cdots y}^{m\text{ times}}) = \langle\langle(\xi_x - \bar{u}_x)^n(\xi_y - \bar{u}_y)^m\varphi_{eq}\rangle\rangle. \tag{4}$$

In particular, taking into account Eq. (1), it is immediate to realize that the first even central moments are

$$\hat{\gamma}^{eq} = \bar{\rho}, \qquad \hat{\gamma}^{eq}_{xx} = \hat{\gamma}^{eq}_{yy} = \bar{\rho}/3, \qquad \hat{\gamma}^{eq}_{xy} = 0, \qquad \hat{\gamma}^{eq}_{xxyy} = \bar{\rho}/9,$$

while the first odd central moments are $\hat{\gamma}^{eq}_x = \hat{\gamma}^{eq}_y = \hat{\gamma}^{eq}_{xxy} = \hat{\gamma}^{eq}_{yyx} = 0$.

B. Discrete velocity space

Concerning the discrete velocity space, let us consider the D2Q9 lattice, where the discrete velocity component v_i has the following values:

$$v_x = \begin{bmatrix} 0, & -1, & -1, & -1, & 0, & 1, & 1, & 1, & 0 \end{bmatrix}^T, \qquad v_y = \begin{bmatrix} 0, & 1, & 0, & -1, & -1, & -1, & 0, & 1, & 1 \end{bmatrix}^T.$$

Before proceeding, let us define the rule of computation for the lists. Let h and g be the lists defined by $h = [h_0, h_1, h_2, \cdots, h_8]^T$ and $g = [g_0, g_1, g_2, \cdots, g_8]^T$. Then, hg is the list defined by $[h_0 g_0, h_1 g_1, h_2 g_2, \cdots, h_8 g_8]^T$. The sum of all the elements of the list h is denoted by $\langle h \rangle = \sum_{i=0}^{8} h_i$.

The equivalent moment space is defined by a transformation matrix, which is not unique. For example, let us consider the *non–orthogonal* transformation matrix

$$M = [1;\ v_x;\ v_y;\ v_x^2;\ v_y^2;\ v_x v_y;\ (v_x)^2 v_y;\ v_x (v_y)^2;\ (v_x)^2 (v_y)^2]^T,$$

which involves proper combinations of the lattice velocity components. The transformation described by the matrix M diagonalizes the collisional operator of the TRT model (see [18], even though this simple property is not clearly stated there). On the other hand, let us define the following *orthogonal* transformation matrix (considered in [19])

$$K = \begin{bmatrix} 1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 4 \\ 1 & -1 & 1 & 2 & 0 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & -2 & -2 \\ 1 & -1 & -1 & 2 & 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & -2 \\ 1 & 1 & -1 & 2 & 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 2 & 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & -2 \end{bmatrix}, \tag{5}$$

where clearly $K^T K$ is diagonal.

The dimensionless density $\bar{\rho}$ and flow velocity $\bar{u}_i$ are defined by $\bar{\rho} = \langle f \rangle$ and $\bar{\rho}\bar{u}_i = \langle v_i f \rangle$, where f is the discrete distribution function. Let us introduce the generic discrete *raw* moment

$$\pi_{xx\cdots x\, yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}}, \overbrace{yy\cdots y}^{m\text{ times}}) = \langle v_x^n v_y^m f \rangle, \tag{6}$$

and the corresponding generic discrete *central* moment

$$\hat{\pi}_{xx\cdots x\, yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}}, \overbrace{yy\cdots y}^{m\text{ times}}) = \langle (v_x - \bar{u}_x)^n (v_y - \bar{u}_y)^m f \rangle. \tag{7}$$

III. CASCADED LBM

The generic LBM algorithm consists of a *collision process* and a *streaming process*. Following [19], we define the *collision process* as

$$f^p = f + K\, g(f, f_{eq}, \lambda_e, \lambda_o), \tag{8}$$

where f_{eq} is the discrete local equilibrium, λ_e and λ_o are the relaxation frequencies for the even and odd moments respectively and f^p is the post–collision distribution function. All the previous quantities are computed in $(\bar{t}, \bar{x}_i, v_i)$, where $\bar{t}$ and $\bar{x}_i$ are the time and space in lattice units respectively. We define the *streaming step* as $f(\bar{t}+1, \bar{x}_i + v_i, v_i) = f^p(\bar{t}, \bar{x}_i, v_i)$.

Because of the collisional invariants, $g_0 = g_1 = g_2 = 0$. Concerning the remaining terms g_α ($\alpha = 3-8$), following [19], let us consider first the particular case $\lambda_e = \lambda_o = 1$, which implies that the post–collision distribution function is in equilibrium, namely

$$f^p_{eq} = f + K\, g^*, \tag{9}$$

where $g^* = g(f, f_{eq}, 1, 1)$. Let us multiply Eq. (9) by $(v_x - \bar{u}_x)^n (v_y - \bar{u}_y)^m$, let us take the sum $\langle\cdot\rangle$ of the resulting list and, finally, let us assume that the equilibrium moments of the post–collision discrete function coincide with the continuous counterparts, namely

$$\langle (v_x - \bar{u}_x)^n (v_y - \bar{u}_y)^m K g^*_\alpha \rangle = \hat{\gamma}^{eq}_{xx\cdots x\, yy\cdots y} - \hat{\pi}_{xx\cdots x\, yy\cdots y}, \tag{10}$$

where $\alpha = 3-8$. In particular, considering the first moments (discussed in Section II) and realizing that the left hand side of Eq. (10) is linear with regards to g^*_α ($\alpha = 3-8$) yields

$$S \begin{bmatrix} g^*_3 \\ g^*_4 \\ g^*_5 \\ g^*_6 \\ g^*_7 \\ g^*_8 \end{bmatrix} = \begin{bmatrix} \hat{\gamma}^{eq}_{xx} - \hat{\pi}_{xx} \\ \hat{\gamma}^{eq}_{yy} - \hat{\pi}_{yy} \\ \hat{\gamma}^{eq}_{xy} - \hat{\pi}_{xy} \\ \hat{\gamma}^{eq}_{xxy} - \hat{\pi}_{xxy} \\ \hat{\gamma}^{eq}_{xyy} - \hat{\pi}_{xyy} \\ \hat{\gamma}^{eq}_{xxyy} - \hat{\pi}_{xxyy} \end{bmatrix}, \tag{11}$$

where S is the shift matrix for passing from the *frame at rest* to the *moving frame*, namely

$$S = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ -6\bar{u}_y & -2\bar{u}_y & 8\bar{u}_x & -4 & 0 & 0 \\ -6\bar{u}_x & 2\bar{u}_x & 8\bar{u}_y & 0 & -4 & 0 \\ 8+6(\bar{u}_x^2+\bar{u}_y^2) & 2(\bar{u}_y^2-\bar{u}_x^2) & -16\bar{u}_x\bar{u}_y & 8\bar{u}_y & 8\bar{u}_x & 4 \end{bmatrix}, \tag{12}$$

while the vector at the right hand side of Eq. (11) is

$$\begin{bmatrix} \hat{\pi}_{xx} \\ \hat{\pi}_{yy} \\ \hat{\pi}_{xy} \\ \hat{\pi}_{xxy} \\ \hat{\pi}_{xyy} \\ \hat{\pi}_{xxyy} \end{bmatrix} = \begin{bmatrix} \pi_{xx} - \bar{\rho}\bar{u}_x^2 \\ \pi_{yy} - \bar{\rho}\bar{u}_y^2 \\ \pi_{xy} - \bar{\rho}\bar{u}_x\bar{u}_y \\ \pi_{xxy} - \pi_{xx}\bar{u}_y - 2\bar{u}_x\pi_{xy} + 2\bar{\rho}\bar{u}_x^2\bar{u}_y \\ \pi_{xyy} - 2\pi_{xy}\bar{u}_y - \bar{u}_x\pi_{yy} + 2\bar{\rho}\bar{u}_x\bar{u}_y^2 \\ \pi_{xxyy} - 2\pi_{xxy}\bar{u}_y - 2\bar{u}_x\pi_{xyy} + \pi_{xx}\bar{u}_y^2 + \bar{u}_x^2\pi_{yy} + 4\bar{u}_x\bar{u}_y\pi_{xy} - 3\bar{\rho}\bar{u}_x^2\bar{u}_y^2 \end{bmatrix}. \tag{13}$$

Solving the system of equations given by Eq. (11) yields

$$\begin{bmatrix} g_3^* \\ g_4^* \\ g_5^* \\ g_6^* \\ g_7^* \\ g_8^* \end{bmatrix} = \begin{bmatrix} -(\pi_{xx}+\pi_{yy})/12 + \bar{\rho}/18 + \bar{\rho}\bar{u}_x^2/12 + \bar{\rho}\bar{u}_y^2/12 \\ -(\pi_{xx}-\pi_{yy})/4 + \bar{\rho}\bar{u}_x^2/4 - \bar{\rho}\bar{u}_y^2/4 \\ \pi_{xy}/4 - \bar{\rho}\bar{u}_x\bar{u}_y/4 \\ \pi_{xxy}/4 - \bar{\rho}\bar{u}_y/12 - \bar{\rho}\bar{u}_x^2\bar{u}_y/4 \\ \pi_{xyy}/4 - \bar{\rho}\bar{u}_x/12 - \bar{\rho}\bar{u}_x\bar{u}_y^2/4 \\ (\pi_{xx}+\pi_{yy})/6 - \pi_{xxyy}/4 - \bar{\rho}/12 - \bar{\rho}\bar{u}_x^2/12 - \bar{\rho}\bar{u}_y^2/12 + \bar{\rho}\bar{u}_x^2\bar{u}_y^2/4 \end{bmatrix}. \tag{14}$$

A. Recovering traditional TRT scheme

Before proceeding with the derivation reported in [19], let consider first the particular choice $g_3 = \lambda_e\, g_3^*$, $g_4 = \lambda_e\, g_4^*$, $g_5 = \lambda_e\, g_5^*$, $g_6 = \lambda_o\, g_6^*$, $g_7 = \lambda_o\, g_7^*$ and $g_8 = \lambda_e\, g_8^*$. In this case, Eq. (8) can be rewritten in a simpler way

$$f^p = f + K\,g = f + M^{-1}(M\,K\,g) = f + A(f_{eq} - f), \tag{15}$$

where $A = M^{-1}\Lambda\, M$,

$$\Lambda = \mathrm{diag}([0,0,0,\lambda_e,\lambda_e,\lambda_e,\lambda_o,\lambda_o,\lambda_e]),$$

and

$$
Mf_{eq} = \begin{bmatrix} \pi_0^{eq} \\ \pi_1^{eq} \\ \pi_2^{eq} \\ \pi_{xx}^{eq} \\ \pi_{yy}^{eq} \\ \pi_{xy}^{eq} \\ \pi_{xxy}^{eq} \\ \pi_{xyy}^{eq} \\ \pi_{xxyy}^{eq} \end{bmatrix} = \begin{bmatrix} \bar{\rho} \\ \bar{\rho}\bar{u}_x \\ \bar{\rho}\bar{u}_y \\ \bar{\rho}/3 + \bar{\rho}\bar{u}_x^2 \\ \bar{\rho}/3 + \bar{\rho}\bar{u}_y^2 \\ \bar{\rho}\bar{u}_x\bar{u}_y \\ \bar{\rho}\bar{u}_y/3 + \bar{\rho}\bar{u}_x^2\bar{u}_y \\ \bar{\rho}\bar{u}_x/3 + \bar{\rho}\bar{u}_x\bar{u}_y^2 \\ \bar{\rho}/9 + \bar{\rho}/3(\bar{u}_x^2 + \bar{u}_y^2) + \bar{\rho}\bar{u}_x^2\bar{u}_y^2 \end{bmatrix} = \begin{bmatrix} \gamma_0^{eq} \\ \gamma_1^{eq} \\ \gamma_2^{eq} \\ \gamma_{xx}^{eq} \\ \gamma_{yy}^{eq} \\ \gamma_{xy}^{eq} \\ \gamma_{xxy}^{eq} \\ \gamma_{xyy}^{eq} \\ \gamma_{xxyy}^{eq} \end{bmatrix}. \tag{16}
$$

The previous expressions are perfectly equivalent to the TRT scheme with $c_s^2 = 1/3$ [18], which has the bulk viscosity equal to the kinematic viscosity (as explained in section 2.1 of [20]). The previous polynomial equilibrium has the same moments of the continuous Maxwellian, given by Eq. (3). It is possible to prove that A is exactly the collisional matrix of the TRT scheme and f_{eq} is the Taylor expansion of the continuous equilibrium given by Eq. (1) for the D2Q9 lattice. If the terms higher then second-order with regards to macroscopic velocity would be neglected, then the previous equilibrium reduces to the standard expression, which is enough for consistency [8].

Hence, if the raw moments are relaxed in the frame at rest as usual and only two relaxation frequencies are considered, the proposed derivation leads to the two–relaxation–time (TRT) collisional operator with proper polynomial equilibrium. (result C).

B. Recovering cascaded LBM scheme

The previous choice of the relaxation process recovering the TRT scheme, can be interpreted in terms of the following definitions of g_α ($\alpha = 3 - 8$)

$$
\begin{bmatrix} g_3/\lambda_e \\ g_4/\lambda_e \\ g_5/\lambda_e \\ g_6/\lambda_o \\ g_7/\lambda_o \\ g_8/\lambda_e \end{bmatrix} = S^{-1} \begin{bmatrix} \hat{\gamma}_{xx}^{eq} - \hat{\pi}_{xx} \\ \hat{\gamma}_{yy}^{eq} - \hat{\pi}_{yy} \\ \hat{\gamma}_{xy}^{eq} - \hat{\pi}_{xy} \\ \hat{\gamma}_{xxy}^{eq} - \hat{\pi}_{xxy} \\ \hat{\gamma}_{xyy}^{eq} - \hat{\pi}_{xyy} \\ \hat{\gamma}_{xxyy}^{eq} - \hat{\pi}_{xxyy} \end{bmatrix}, \tag{17}
$$

where S^{-1} is the shift matrix for passing from *moving frame* to the *frame at rest*. Clearly in the previous expression, the relaxation is done in the frame at rest. In order to relax the central moments, i.e. the moments shifted by the macroscopic velocity, in the moving frame, it is enough to apply the relaxation frequencies *before* multiplying by S^{-1}.

Actually, in Ref. [19], the relaxation is done neither in the frame at rest nor in the moving frame, but the cascaded relaxation is defined instead. First of all, the particular choice $g'_3 = \lambda_e^\xi\, g_3^*$, $g'_4 = \lambda_e^\nu\, g_4^*$, $g'_5 = \lambda_e^\nu\, g_5^*$ is assumed (which is equivalent to relax the stress tensor components in the frame at rest), where λ_e^ν is the relaxation frequency controlling the kinematic viscosity and λ_e^ξ is that controlling the bulk viscosity. By means of the forth and fifth rows of matrix S defined by Eq. (12), the quantities g'_6 and g'_7 are computed, namely

$$-6\bar{u}_y g'_3 - 2\bar{u}_y g'_4 + 8\bar{u}_x g'_5 - 4g'_6 \;=\; \lambda_o(\hat{\gamma}^{eq}_{xxy} - \hat{\pi}_{xxy}), \tag{18}$$

$$-6\bar{u}_x g'_3 + 2\bar{u}_x g'_4 + 8\bar{u}_y g'_5 - 4g'_7 \;=\; \lambda_o(\hat{\gamma}^{eq}_{xyy} - \hat{\pi}_{xyy}), \tag{19}$$

and, by means of the last row of matrix S, the quantity the g'_8 is computed, namely

$$\left[8 + 6(\bar{u}_x^2 + \bar{u}_y^2)\right] g'_3 + 2(\bar{u}_y^2 - \bar{u}_x^2)g'_4 - 16\bar{u}_x\bar{u}_y g'_5 + 8\bar{u}_y g'_6 + 8\bar{u}_x g'_7 + 4g'_8 = \lambda_e(\hat{\gamma}^{eq}_{xxyy} - \hat{\pi}_{xxyy}) \tag{20}$$

The previous choice is equivalent to relax in the moving frame the higher order moments.

Also in this case, it is possible to search for a simplified evolution equation, namely

$$f'^p = f + K\, g' = f + M^{-1}(M\, K\, g') = f + A'(f'_{eq} - f), \tag{21}$$

where $A' = M^{-1}\Lambda'\, M$ and Λ' is the block–diagonal matrix defined as

$$\Lambda' = \mathrm{diag}\left([0,0,0], \begin{bmatrix} \lambda_e^+ & \lambda_e^- \\ \lambda_e^- & \lambda_e^+ \end{bmatrix}, [\lambda_e^\nu, \lambda_o, \lambda_o, \lambda_e] \right), \tag{22}$$

where $\lambda_e^+ = (\lambda_e^\xi + \lambda_e^\nu)/2$ and $\lambda_e^- = (\lambda_e^\xi - \lambda_e^\nu)/2$, while the moments of f'_{eq} are identical to those of f_{eq} reported in Eq. (16), with the exception of

$$\begin{aligned}\pi'^{eq}_{xxy} \;=\;& \pi^{eq}_{xxy} + (1-\omega_\xi)/2\,\bar{u}_y\left[(\pi_{xx} - \pi^{eq}_{xx}) + (\pi_{yy} - \pi^{eq}_{yy})\right] \\ &+ (1-\omega_\nu)/2\,\bar{u}_y\left[(\pi_{xx} - \pi^{eq}_{xx}) - (\pi_{yy} - \pi^{eq}_{yy})\right] + 2\,(1-\omega_\nu)\,\bar{u}_x(\pi_{xy} - \pi^{eq}_{xy}),\end{aligned} \tag{23}$$

$$\begin{aligned}\pi'^{eq}_{xyy} \;=\;& \pi^{eq}_{xyy} + (1-\omega_\xi)/2\,\bar{u}_x\left[(\pi_{yy} - \pi^{eq}_{yy}) + (\pi_{xx} - \pi^{eq}_{xx})\right] \\ &+ (1-\omega_\nu)/2\,\bar{u}_x\left[(\pi_{yy} - \pi^{eq}_{yy}) - (\pi_{xx} - \pi^{eq}_{xx})\right] + 2\,(1-\omega_\nu)\,(\pi_{xy} - \pi^{eq}_{xy})\,\bar{u}_y,\end{aligned} \tag{24}$$

$$
\begin{aligned}
\pi'^{eq}_{xxyy} &= \pi^{eq}_{xxyy} + 2\,(1-\theta)\left[\bar{u}_x(\pi_{xyy} - \pi^{eq}_{xyy}) + (\pi_{xxy} - \pi^{eq}_{xyy})\,\bar{u}_y\right] \\
&\quad -2\,(1-\theta)\left[\bar{u}_x^2(\pi_{yy} - \pi^{eq}_{yy}) + (\pi_{xx} - \pi^{eq}_{xx})\,\bar{u}_y^2 + 4\,\bar{u}_x\bar{u}_y(\pi_{xy} - \pi^{eq}_{xy})\right] \\
&\quad +(1-\theta_\xi)/2\ \left[(\bar{u}_x^2 + \bar{u}_y^2)\left((\pi_{yy} - \pi^{eq}_{yy}) + (\pi_{xx} - \pi^{eq}_{xx})\right)\right] \\
&\quad +(1-\theta_\nu)/2\ \left[(\bar{u}_x^2 - \bar{u}_y^2)\left((\pi_{yy} - \pi^{eq}_{yy}) - (\pi_{xx} - \pi^{eq}_{xx})\right)\right] \\
&\quad +4\,(1-\theta_\nu)\bar{u}_x\bar{u}_y(\pi_{xy} - \pi^{eq}_{xy}),
\end{aligned} \tag{25}
$$

where $\omega_\nu = \lambda_e^\nu/\lambda_o$, $\omega_\xi = \lambda_e^\xi/\lambda_o$, $\theta = \lambda_o/\lambda_e$, $\theta_\nu = \lambda_e^\nu/\lambda_e$ and $\theta_\xi = \lambda_e^\xi/\lambda_e$. Clearly in case of single relaxation time, $\omega_\nu = \omega_\xi = \theta = \theta_\nu = \theta_\xi = 1$ and $f'_{eq} = f_{eq}$, proving that, for the BGK scheme [1], the cascaded relaxation coincides with the relaxation of the raw moments in the frame at rest. However in general, relaxing differently the central moments in the moving frame, is equivalent to consider a *generalized local equilibrium*, depending on both conserved (as it happens in kinetic theory) and non–conserved moments, such as π_{ij} and π_{ijk}, in the frame at rest (result A). Clearly the vice versa holds as well, because relaxing differently the moments in the frame at rest (as usual) leads to a generalization of the equilibrium in the moving frame. Hence the previous result seems to suggest that, among all the possible relaxations which can be recasted in the form given by Eqs. (21–25), only the BGK relaxation actually avoids any equilibrium generalization in any frame.

IV. GRAD MOMENT EXPANSION

In order to check that the numerical scheme is actually consistent with regards to the desired incompressible Navier–Stokes equations, let us apply the procedure proposed in Ref. [21], based on the Grad moment expansion.

Let us introduce first the diffusion scaling [7, 8]. Introducing the small parameter ϵ as $\epsilon = l_c/L$, which corresponds to the Knudsen number, where l_c is the mean free path and L is a macroscopic characteristic length, we have $x_i = \epsilon \bar{x}_i$. Furthermore assuming $U/c = \epsilon$, which corresponds to the Mach number, where U is the macroscopic characteristic speed and c is proportional to the sound speed, we have $t = \epsilon^2 \bar{t}$. Consequently, plugging the collisional operator given by Eq. (21) in a kinetic evolution equation for f yields

$$
\epsilon^2 \frac{\partial f}{\partial t} + \epsilon v_i \frac{\partial f}{\partial x_i} = A' \left(f'_{eq} - f\right). \tag{26}
$$

Taking into account that $\bar{u}_i = \epsilon\, u_i$ because of the considered low Mach number limit, let us compute the first moments of the Eq. (26), namely

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} u_i)}{\partial x_i} = 0, \tag{27}$$

$$\epsilon^3 \frac{\partial (\bar{\rho} u_i)}{\partial t} + \epsilon \frac{\partial \pi_{ij}}{\partial x_j} = 0, \tag{28}$$

where the stress tensor components satisfy

$$\epsilon^2 \frac{\partial \pi_{xx}}{\partial t} + \epsilon \frac{\partial \pi_{xx\,k}}{\partial x_k} = \lambda_\sigma^+ \left(\pi_{xx}^{eq} - \pi_{xx}\right) + \lambda_\sigma^- \left(\pi_{yy}^{eq} - \pi_{yy}\right), \tag{29}$$

$$\epsilon^2 \frac{\partial \pi_{yy}}{\partial t} + \epsilon \frac{\partial \pi_{yy\,k}}{\partial x_k} = \lambda_\sigma^- \left(\pi_{xx}^{eq} - \pi_{xx}\right) + \lambda_\sigma^+ \left(\pi_{yy}^{eq} - \pi_{yy}\right), \tag{30}$$

$$\epsilon^2 \frac{\partial \pi_{xy}}{\partial t} + \epsilon \frac{\partial \pi_{xy\,k}}{\partial x_k} = \lambda_\sigma^\nu \left(\pi_{xy}^{eq} - \pi_{xy}\right), \tag{31}$$

Since $O(\pi_{ijk}) = O(\pi_{ijk}^{eq}) = \epsilon$ [21], the previous equations prove that $O(\pi_{ij} - \pi_{ij}^{eq}) = O(\epsilon^2)$. Introducing this result in Eqs. (23, 24) yields

$$\pi_{xxy}'^{eq} - \pi_{xxy}^{eq} = O(\epsilon^3), \qquad \pi_{xyy}'^{eq} - \pi_{xyy}^{eq} = O(\epsilon^3).$$

Searching for approximated expressions of the stress tensor components, it is possible to assume that $\pi_{xxy} \sim \pi_{xxy}'^{eq} \sim \pi_{xxy}^{eq}$ and $\pi_{xyy} \sim \pi_{xyy}'^{eq} \sim \pi_{xyy}^{eq}$, without affecting the second order accuracy of the method. The generalized local equilibrium differs from the Taylor–expansion–based equilibrium given by Eq. (16) for higher–order terms, which do not modify the recovered macroscopic equations up to the incompressible Navier–Stokes level (result B).

V. CONCLUSIONS

Cascaded LBM [19] represents a new approach in order to enhance the stability of the traditional MRT–LBM schemes. The present work shows that the cascaded LBM uses a generalized local equilibrium in the frame at rest, which depends on both conserved and non–conserved moments. This new equilibrium does not affect the consistency of LBM. These results may clarify the essence of the cascaded LBM and they may help in developing new schemes in a systematic way.

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