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Nontrivial Solutions of $p$-Superlinear $p$-Laplacian Problems via a Cohomological Local Splitting

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Abstract

We consider a quasilinear equation, involving the p-Laplace operator, with a p-superlinear nonlinearity. We prove the existence of a nontrivial solution, also when there is no mountain pass geometry, without imposing a global sign condition. Techniques of Morse theory are employed.

Keywords: p-Laplace equations; nontrivial solutions; Morse theory.

Mathematics Subject Classification 2010: 58E05, 35J65

1 Introduction

Consider the boundary value problem

\[
\begin{aligned}
-\Delta_p u &= \lambda V |u|^{p-2} u + g(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), \( n \geq 1 \), \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the p-Laplacian of \( u \), \( p \in ]1, \infty[ \), \( \lambda \in \mathbb{R} \) is a parameter, \( V \in L^\infty(\Omega) \) and \( g \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying the following conditions:

\((g1)\) there exist \( C > 0 \) and

\[
p < q < \begin{cases} p^* := \frac{np}{n-p} & \text{if } p < n \\ \infty & \text{if } p \geq n \end{cases}
\]

such that

\[ |g(x, s)| \leq C (|s|^{q-1} + 1) ; \]

\((g2)\) we have

\[ g(x, s) = o(|s|^{p-1}) \text{ as } s \to 0, \text{ uniformly in } x; \]
(g3) there exist $\mu > p$ and $R > 0$ such that

$$0 < \mu G(x, s) := \int_0^s g(x, t) \, dt \leq s \, g(x, s), \quad \text{whenever } |s| \geq R.$$ 

In particular, $g(x, 0) \equiv 0$ and hence we have the trivial solution $u = 0$, and we seek another.

In the case $p = 2$, the existence of a nontrivial solution $u$ for (1.1) can be obtained via the Linking Theorem (see e.g. Rabinowitz [21, Theorem 5.16]). More precisely, let us assume, without loss of generality, that $\lambda \geq 0$. If the set

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} V |u|^p \, dx = 1 \right\}$$

is empty or if $\mathcal{M} \neq \emptyset$ and

$$\lambda < \lambda_1 := \min \left\{ \int_{\Omega} |\nabla u|^p \, dx : \ u \in \mathcal{M} \right\},$$

then the existence of a nontrivial solution can be proved, without any further assumption, by the Mountain Pass Theorem for any $p > 1$ (see Ambrosetti and Rabinowitz [1] for the case $p = 2$ and Dinca, Jebelean and Mawhin [10] for the case $p \neq 2$). On the contrary, if $\mathcal{M} \neq \emptyset$ and $\lambda \geq \lambda_1$, the classical proof is based on the fact that each eigenvalue $\lambda_k$ of $-\Delta_2$ induces a suitable direct sum decomposition of $W_0^{1,2}(\Omega)$. On the other hand, if $p \neq 2$, such decompositions are not available. Nevertheless, a linking argument over cones, rather than over linear subspaces, has been developed for $p \neq 2$, when $\lambda$ is close to $\lambda_1$ by Fan and Z. Li [12] and for any $\lambda$ by Degiovanni and Lancelotti [9]. In such a way, the mentioned result of Rabinowitz has been completely extended to the case $p \neq 2$.

When $\lambda \geq \lambda_1$, in all these results a global sign condition like $G(x, s) \geq 0$ needs to be imposed, in order to recognize the linking geometry. However, such an assumption can be relaxed by means of Morse theory or nonstandard linking constructions.

When $p = 2$, in Benci [2, Theorem 7.14] it is show that, if a nonresonance condition at the origin is satisfied, the existence of a nontrivial solution can be obtained without any further assumption. On the other hand, S.J. Li and Willem [15, Theorem 4] are able to treat the resonant case under a local sign condition on $G$. Related results are also contained in J.Q. Liu and S.J. Li [14].
The approach based on Morse theory has been extended to the case \( p \neq 2 \) by S. Liu [16] when \( \lambda \) is close to \( \lambda_1 \) and by Perera [19] when \( \lambda \) does not belong to the spectrum of the \( p \)-Laplace operator.

Our purpose is to develop this approach, in order to remove any condition on \( \lambda \) and require only a local sign condition on \( G \). Our result is the following

**Theorem 1.1.** Let us suppose that assumptions \((g1)-(g3)\) hold and let \( V \in L^\infty(\Omega) \). Then, for every \( \lambda \in \mathbb{R} \), problem (1.1) has a nontrivial solution \( u \in W^{1,p}_0(\Omega) \) in each of the following cases:

(a) there exists \( \delta > 0 \) such that \( G(x,s) \geq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \);

(b) there exists \( \delta > 0 \) such that \( G(x,s) \leq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \).

This is a natural extension to the case \( p \neq 2 \) of the mentioned result of S.J. Li and Willem, although the argument is based there on a nonstandard linking construction and here on Morse theory.

In the next section we recall and prove some preliminary facts, while in section 3 we prove the main result in a more general setting. In the last section we recover Theorem 1.1 as a particular case.

## 2 Preliminaries

Let \( \Phi \) be a \( C^1 \)-functional defined on a real Banach space \( W \). We denote by \( B_\rho \) and \( S_\rho \) the closed ball and sphere of center 0 and radius \( \rho \). We also denote by \( H \) the Alexander-Spanier cohomology with \( \mathbb{Z}_2 \)-coefficients (see Spanier [22]). For a symmetric subset \( X \) of \( W \setminus \{0\} \), \( i(X) \) denotes its \( \mathbb{Z}_2 \)-cohomological index (see Fadell and Rabinowitz [11]). The following notion has been introduced, in a slightly different form, by Perera, Agarwal, and O’Regan [20] and is in turn a variant of the homological local linking of Perera [18]. It should also be compared with the local linking of S.J. Li and Willem [15].

**Definition 2.1.** We say that \( \Phi \) has a cohomological local splitting near 0 in dimension \( k < \infty \), if there are two symmetric cones \( W_-, W_+ \) in \( W \) and \( \rho > 0 \) such that

\[
W_- \cap W_+ = \{0\} , \quad i(W_- \setminus \{0\}) = i(W \setminus W_+) = k
\]  

(2.1)
and
\[
\begin{aligned}
\Phi(u) &\leq \Phi(0) \quad \text{for every } u \in B_\rho \cap W_-, \\
\Phi(u) &\geq \Phi(0) \quad \text{for every } u \in B_\rho \cap W_+.
\end{aligned}
\tag{2.2}
\]

As we will see, in such a case 0 must be a critical point of \( \Phi \).

Recall that the cohomological critical groups of \( \Phi \) at a point \( u \in W \) are defined by
\[
C^q(\Phi, u) = H^q(\Phi^c, \Phi^c \setminus \{u\}), \quad q \geq 0,
\]
where \( c = \Phi(u) \) is the corresponding value and \( \Phi^c \) is the closed sublevel set \( \{w \in W : \Phi(w) \leq c\} \) (see, e.g., Chang [3] or Mawhin and Willem [17]). By the excision property, we have
\[
C^q(\Phi, u) \approx H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\})
\]
for every neighborhood \( U \) of \( u \). Therefore, the concept has local nature.

Moreover, it is well known that all critical groups are trivial, if \( u \) is not a critical point of \( \Phi \) (see e.g. Corvellec [5, Proposition 3.4]). Finally, the next result shows a stability property and is a particular case of Corvellec and Hantoute [7, Theorem 5.2] (see also Benci [2, Theorem 5.16]). The case in which \( W \) is a Hilbert space and \( \Phi \) is of class \( C^2 \) can be also found in Chang [3, Theorem I.5.6] and in Mawhin and Willem [17, Theorem 8.8].

**Theorem 2.2.** Let \( \Phi_t : W \rightarrow \mathbb{R}, \ t \in [0,1], \) be a family of functionals of class \( C^1 \). Assume that there exists \( \rho > 0 \) such that each \( \Phi_t \) satisfies the Palais-Smale condition over \( B_\rho \) and has no critical point in \( B_\rho \) other than \( 0 \). Suppose also that the map \( \{t \mapsto \Phi_t\} \) is continuous from \([0,1]\) into \( C^1(B_\rho) \).

Then \( C^q(\Phi_t, 0) \) is independent of \( t \).

The cohomological local splitting allows to give an estimate of the critical groups, also in the absence of a direct sum decomposition.

**Proposition 2.3.** If \( \Phi \) has a cohomological local splitting near \( 0 \) in dimension \( k \), then \( 0 \) is a critical point of \( \Phi \). Moreover, if \( 0 \) is an isolated critical point of \( \Phi \), then we have \( C^k(\Phi, 0) \neq 0 \).
This proposition is a variant of a result of Perera, Agarwal, and O’Regan [20]. We need the following lemma from Degiovanni and Lancelotti (see [9, Theorem 2.7] and also Cingolani and Degiovanni [4, Theorem 3.6]), which establishes a connection between equivariant index and nonequivariant cohomology.

**Lemma 2.4.** If $X$ is a symmetric subset of $W \setminus \{0\}$ with $k = i(X) < \infty$ and $A$ is a symmetric subset of $X$ with $i(A) = k$, then the homomorphism $i^* : H^k(W, X) \to H^k(W, A)$, induced by the inclusion $i : (W, A) \subseteq (W, X)$, is nontrivial.

**Proof of Proposition 2.3.** It is enough to prove that, if 0 does not accumulate critical points of $\Phi$, then $C_k(\Phi; 0) \neq 0$. Therefore assume, without loss of generality, that $\Phi$ has no critical point $u$ with $0 < \|u\| \leq \rho$. Let $c = \Phi(0)$.

There exists a deformation $\eta : W \times [0, 1] \to W$ such that

\[
\Phi(\eta(u, t)) < \Phi(u) \quad \text{if } \Phi'(u) \neq 0 \text{ and } t > 0,
\]

\[
\eta(u, t) = u \quad \text{otherwise},
\]

(see e.g. Benci [2, Theorem 5.5] or Corvellec [6]). Let $0 < r \leq \rho$ be such that $\eta(B_r \times [0, r]) \subseteq B_{r'}$. Since $B_r \cap W_-$ is contractible and $S_r \cap W_-$ is a deformation retract of $W_- \setminus \{0\}$, from Lemma 2.4 and (2.1) we deduce that the homomorphism

\[
i^* : H^k(W, B_r \setminus W_+) \to H^k(B_r \cap W_-, S_r \cap W_-),
\]

induced by the inclusion $i : (B_r \cap W_-, S_r \cap W_-) \subseteq (W, B_r \setminus W_+)$, is nontrivial. On the other hand, since (2.2) implies

\[
B_r \cap W_- \subseteq \Phi^c \cap B_r, \quad S_r \cap W_- \subseteq \Phi^c \cap B_r \setminus \{0\}, \quad \eta(\Phi^c \cap B_r \setminus \{0\}, r) \subseteq B_{r'} \setminus W_+,
\]

we may also consider the composition

\[
H^k(W, B_{r'} \setminus W_+) \xrightarrow{\eta(\cdot, r)^*} H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \xrightarrow{j^*} H^k(B_r \cap W_-, S_r \cap W_-)
\]

where $j : (B_r \cap W_-, S_r \cap W_-) \subseteq (\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\})$ is the inclusion. Again (2.2) yields

\[
\eta((S_r \cap W_-) \times [0, r]) \subseteq B_{r'} \setminus W_+,
\]

so that $\eta(\cdot, r) \circ j$ is homotopic to $i$. Therefore $j^* \circ \eta(\cdot, r)^* = i^*$ is nontrivial, which in turn implies that $H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \neq 0$. \qed
Now, let us recall a situation in which one can build two symmetric cones satisfying (2.1). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), let \( 1 < p < \infty \) and let

\[
\mathcal{V}(\Omega) := \left\{ \begin{array}{ll}
L^r(\Omega) & \text{if } p \leq n, \\
L^1(\Omega) & \text{if } p > n.
\end{array} \right.
\]

Take \( V \in \mathcal{V}(\Omega) \) and consider the eigenvalue problem

\[
\begin{cases}
-\Delta_p u = \lambda V |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{2.3}
\]

We refer the reader to Cuesta [8] and Szulkin and Willem [23] for general properties concerning (2.3).

Now, assume that \( \{ x \in \Omega : V(x) > 0 \} \) has positive measure, denote by \( \mathcal{F} \) the class of symmetric subsets of

\[
\mathcal{M} = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} V |u|^p \, dx = 1 \right\}
\]

and set

\[
\lambda_k = \inf_{M \in \mathcal{F}} \sup_{u \in M, i(M) \geq k} \int_{\Omega} |\nabla u|^p \, dx, \quad k \geq 1.
\tag{2.4}
\]

Then \( \lambda_k \nearrow +\infty \) are eigenvalues of (2.3) and the following result holds (see Degiovanni and Lancelotti [9, Theorem 3.2]).

**Proposition 2.5.** Let \( k \geq 1 \) be such that \( \lambda_k < \lambda_{k+1} \) and let

\[
W_- = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u|^p \, dx \leq \lambda_k \int_{\Omega} V |u|^p \, dx \right\},
\]

\[
W_+ = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u|^p \, dx \geq \lambda_{k+1} \int_{\Omega} V |u|^p \, dx \right\}.
\]

Then \( W_- \) and \( W_+ \) are two symmetric cones in \( W^{1,p}_0(\Omega) \) satisfying (2.1).
3 The main result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$, let $V \in \mathcal{V}(\Omega)$ and let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the following assumptions:

$(g1')$ we have that

for every $\varepsilon > 0$ there exists $a_\varepsilon \in \mathcal{V}(\Omega)$ such that

$$|g(x, s)| \leq a_\varepsilon(x) |s|^{p-1} + \varepsilon |s|^{p^*-1},$$

if $p < n$;

there exist $a \in \mathcal{V}(\Omega)$, $C > 0$ and $q > p$ such that

$$|g(x, s)| \leq a(x) |s|^{p-1} + C |s|^{q-1},$$

if $p = n$;

for every $S > 0$ there exists $a_S \in \mathcal{V}(\Omega)$ such that

$$|g(x, s)| \leq a_S(x) |s|^{p-1}$$

whenever $|s| \leq S$, if $p > n$;

$(g2')$ for a.e. $x \in \Omega$, we have $\lim_{s \rightarrow 0} \frac{G(x, s)}{|s|^p} = 0$ and $\lim_{|s| \rightarrow \infty} \frac{G(x, s)}{|s|^p} = +\infty$, where

$$G(x, s) = \int_0^s g(x, t) \, dt;$$

$(g3')$ there exist $\mu > p$, $\gamma_0 \in L^1(\Omega)$ and $\gamma_1 \in \mathcal{V}(\Omega)$ such that

$$-\gamma_0(x) - \gamma_1(x) |s|^p \leq \mu G(x, s) \leq s g(x, s) + \gamma_0(x) + \gamma_1(x) |s|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

In order to study the quasilinear problem

$$\begin{cases}
-\Delta_p u = \lambda V |u|^{p-2} u + g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \tag{3.1}$$

let us define a functional $\Phi : W^{1,p}_0(\Omega) \rightarrow \mathbb{R}$ of class $C^1$ by

$$\Phi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \int_\Omega V |u|^p \, dx - \int_\Omega G(x, u) \, dx$$

and set $\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$ for every $u \in W^{1,p}_0(\Omega)$. Recall also that, for every $\gamma \in \mathcal{V}(\Omega)$, the map $\{u \mapsto \gamma |u|^p\}$ is weak-to-strong sequentially continuous from $W^{1,p}_0(\Omega)$ into $L^1(\Omega)$. 

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Lemma 3.1. The following facts hold:

(a) for every \( c \in \mathbb{R} \), we have
\[
\limsup_{\|u\| \to \infty \atop \Phi(u) \leq c} \frac{\Phi'(u)u - p \Phi(u)}{\|u\|^p} < 0;
\]

(b) for every \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \), we have
\[
\lim_{|t| \to \infty} \frac{\Phi(tu)}{|t|^p} = -\infty.
\]

Proof. (a) Let \( c \in \mathbb{R} \). By contradiction, let \( d_k \to 0 \) and let \((u_k)\) be a sequence in \( \Phi^c \) such that \( \|u_k\| \to \infty \) and
\[
\Phi'(u_k)u_k - p \Phi(u_k) \geq -d_k \|u_k\|^p \quad \text{for every } k \in \mathbb{N}.
\]
If we set \( v_k = u_k/\|u_k\| \), up to a subsequence \((v_k)\) is convergent to some \( v \in W_0^{1,p}(\Omega) \) weakly and a.e. in \( \Omega \).

From \((g3')\) it follows that
\[
-d_k \|u_k\|^p \leq \Phi'(u_k)u_k - p \Phi(u_k) = \int_\Omega (pG(x, u_k) - u_k g(x, u_k)) \, dx
\]
\[
= \int_\Omega (\mu G(x, u_k) - u_k g(x, u_k)) \, dx - (\mu - p) \int_\Omega G(x, u_k) \, dx
\]
\[
\leq \int_\Omega (\gamma_0 + \gamma_1 |u_k|^p) \, dx - (\mu - p) \int_\Omega G(x, u_k) \, dx,
\]
whence
\[
(\mu - p) \int_\Omega G(x, u_k) \, dx \leq d_k \|u_k\|^p + \int_\Omega (\gamma_0 + \gamma_1 |u_k|^p) \, dx.
\]
(3.2)

Therefore we have
\[
\limsup_k \frac{\int_\Omega G(x, u_k) \, dx}{\|u_k\|^p} < +\infty.
\]
(3.3)

On the other hand, \((g3')\) also yields
\[
\frac{G(x, u_k)}{\|u_k\|^p} + \frac{\gamma_0}{\|u_k\|^p} \geq -\gamma_1 |v_k|^p,
\]
hence, by the (generalized) Fatou lemma and (3.3),
\[
\int_{\Omega} \left( \liminf_{k} \frac{G(x, u_k)}{\|u_k\|^p} \right) dx \leq \liminf_{k} \frac{\int_{\Omega} G(x, u_k) dx}{\|u_k\|^p} < +\infty.
\]
Since by \((g2')\) we have
\[
\lim_{k} \frac{G(x, u_k)}{\|u_k\|^p} = \lim_{k} \left( \frac{G(x, u_k)}{|u_k|^p} - \frac{v_k}{p} \right) = +\infty \quad \text{where } v \neq 0,
\]
we deduce that \(v = 0\) a.e. in \(\Omega\).

Formula (3.2) can also be rewritten as
\[
\left( \frac{\mu}{p} - 1 \right) \int_{\Omega} |\nabla u_k|^p dx \leq (\mu - p)\Phi(u_k) + d_k\|u_k\|^p + \int_{\Omega} \gamma_0 dx + \int_{\Omega} \left[ \left( \frac{\mu}{p} - 1 \right) V + \gamma_1 \right] |u_k|^p dx,
\]
namely
\[
\left( \frac{\mu}{p} - 1 \right) \leq d_k + (\mu - p)\frac{\Phi(u_k)}{\|u_k\|^p} + \int_{\Omega} \gamma_0 dx + \int_{\Omega} \left[ \left( \frac{\mu}{p} - 1 \right) V + \gamma_1 \right] |v_k|^p dx.
\]

Going to the limit as \(k \to \infty\), we get \(\frac{\mu}{p} - 1 \leq 0\) and a contradiction follows.

(b) Since by \((g3')\)
\[
\frac{G(x, tu)}{|t|^p} + \frac{\gamma_0}{|t|^p} \geq -\gamma_1 |t|^p,
\]
applying as before Fatou's lemma, the assertion follows.

\[\square\]

**Lemma 3.2.** There exists \(a < 0\) such that \(\Phi^a\) is contractible in itself.

**Proof.** By (a) of Lemma 3.1, there exists \(b \in \mathbb{R}\) such that
\[
\Phi'(u)u - p\Phi(u) \leq b \quad \text{for every } u \in \Phi^0.
\]

In particular, there exists \(a < 0\) such that
\[
\Phi'(u)u < 0 \quad \text{for every } u \in \Phi^a. \quad (3.4)
\]
If we set, taking into account \( b \) of Lemma 3.1,
\[
t(u) = \min \{ t \geq 1 : \Phi(tu) \leq a \} ,
\]
from (3.4) we deduce that the function \( \{ u \mapsto t(u) \} \) is continuous. Then \( r(u) = t(u)u \) is a retraction of \( W^{1,p}_0(\Omega) \setminus \{ 0 \} \) onto \( \Phi^a \). Since \( W^{1,p}_0(\Omega) \setminus \{ 0 \} \) is contractible in itself, the same is true for \( \Phi^a \). \( \square \)

**Lemma 3.3.** Assume that 0 is an isolated critical point of \( \Phi \). Then the following facts hold:

(a) if there exists \( \delta > 0 \) such that \( G(x,s) \geq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \), we have \( C^q(\Phi,0) \neq 0 \) for
\[
q = i \left( \left\{ u \in W^{1,p}_0(\Omega) \setminus \{ 0 \} : \int_{\Omega} |\nabla u|^p \, dx \leq \lambda \int_{\Omega} V |u|^p \, dx \right\} \right) ;
\]

(b) if there exists \( \delta > 0 \) such that \( G(x,s) \leq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \), we have \( C^q(\Phi,0) \neq 0 \) for
\[
q = i \left( \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u|^p \, dx < \lambda \int_{\Omega} V |u|^p \, dx \right\} \right) .
\]

**Proof.** By replacing \( (\lambda,V) \) with \( (-\lambda,-V) \) if necessary, we may assume that \( \lambda \geq 0 \). Let \( \vartheta : \mathbb{R} \rightarrow [0,1] \) be a \( C^\infty \)-function such that \( \vartheta(s) = 0 \) for \( |s| \leq \delta / 2 \) and \( \vartheta(s) = 1 \) for \( |s| \geq \delta \). For every \( t \in [0,1] \), define
\[
G_t(x,s) = G(x,(1-t\vartheta(s))s)
\]
and \( \Phi_t : W^{1,p}_0(\Omega) \rightarrow \mathbb{R} \) by
\[
\Phi_t(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} V |u|^p \, dx - \int_{\Omega} G_t(x,u) \, dx .
\]
From \((g1')\) it follows that each \( \Phi_t \) satisfies the Palais-Smale condition over every bounded subset of \( W^{1,p}_0(\Omega) \) (see also [9, Proposition 4.3]). Moreover the map \( \{ t \mapsto \Phi_t \} \) is continuous from \([0,1]\) into \( C^1(B) \) for every bounded subset \( B \) of \( W^{1,p}_0(\Omega) \). We claim that there exists \( \rho > 0 \) such that each \( \Phi_t \) has no critical point in \( B_\rho \) other than 0. By contradiction, let \( (t_j) \) be a sequence in \([0,1]\) and \( (u_j) \) a sequence convergent to 0 with \( \Phi_{t_j}'(u_j) = 0 \).
and \( u_j \neq 0 \). Then the same regularity argument of Guedda and Veron [13, Propositions 1.2 and 1.3] shows that \((u_j)\) is convergent to 0 also in \( L^\infty(\Omega) \). Therefore we have \( \Phi'(u_j) = 0 \) eventually as \( j \to \infty \). Since 0 is an isolated critical point of \( \Phi \), a contradiction follows. From Theorem 2.2 we deduce that \( C^q(\Phi,0) \approx C^q(\Phi_1,0) \) for any \( q \geq 0 \).

Observe also that

\[
\int_\Omega G_1(x,u) \, dx = o(\|u\|^p) \quad \text{as} \quad \|u\| \to 0 \tag{3.5}
\]

(see, e.g., [9, Proposition 4.3]).

In case \((a)\), we have \( G_1(x,s) \geq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \). If the set \( \{ x \in \Omega : V(x) > 0 \} \) has positive measure and \( \lambda \geq \lambda_1 \), take \( k \geq 1 \) such that \( \lambda_k \leq \lambda < \lambda_{k+1} \) and define \( W_-, W_+ \) as in Proposition 2.5. Otherwise, let \( W_- = \{0\} \) and \( W_+ = W_1^{1,p}(\Omega) \).

In any case, \( W_-, W_+ \) are two symmetric cones in \( W_1^{1,p}(\Omega) \) satisfying (2.1) and

\[
i(W_- \setminus \{0\})
= i \left( \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : \int_\Omega |\nabla u|^p \, dx \leq \lambda \int_\Omega V |u|^p \, dx \right\} \right), \tag{3.6}
\]

\[
\int_\Omega |\nabla u|^p \, dx \leq \lambda \int_\Omega V |u|^p \, dx \quad \forall u \in W_-, \tag{3.7}
\]

\[
\exists \sigma \in ]0,1[, \quad (1 - \sigma) \int_\Omega |\nabla u|^p \, dx \geq \lambda \int_\Omega V |u|^p \, dx \quad \forall u \in W_+. \tag{3.8}
\]

From (3.7) and the sign information on \( G_1 \), it follows

\[
\Phi_1(u) \leq 0 \quad \text{for every} \quad u \in W_- \quad \text{with} \quad \|u\| \leq \rho \tag{3.9}
\]

for any \( \rho > 0 \). On the other hand, combining (3.5) and (3.8), we get

\[
\Phi_1(u) \geq 0 \quad \text{for every} \quad u \in W_+ \quad \text{with} \quad \|u\| \leq \rho \tag{3.10}
\]

provided that \( \rho \) is sufficiently small. Therefore also (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.6).

In case \((b)\), we have \( G_1(x,s) \leq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \). If the set \( \{ x \in \Omega : V(x) > 0 \} \) has positive measure and \( \lambda > \lambda_1 \), take now
\[ \lambda_k < \lambda \leq \lambda_{k+1} \] and define \( W_- \), \( W_+ \) as in Proposition 2.5. Otherwise, let \( W_- = \{0\} \) and \( W_+ = W_0^{1,p}(\Omega) \). In any case, \( W_- \), \( W_+ \) are two symmetric cones in \( W_0^{1,p}(\Omega) \) satisfying (2.1) and

\[
i(W_0^{1,p}(\Omega) \setminus W_+)
= i \left( \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p \, dx < \lambda \int_{\Omega} V |u|^p \, dx \right\} \right), \tag{3.11}
\]

\[
\exists \sigma \in [0,1]: (1 + \sigma) \int_{\Omega} |\nabla u|^p \, dx \leq \lambda \int_{\Omega} V |u|^p \, dx \quad \forall u \in W_- \tag{3.12}\]

\[
\int_{\Omega} |\nabla u|^p \, dx \geq \lambda \int_{\Omega} V |u|^p \, dx \quad \forall u \in W_+. \tag{3.13}
\]

Combining (3.5) with (3.12), we get again (3.9) if \( \rho \) is sufficiently small. On the other hand, from (3.13) and the sign information on \( G_1 \) we deduce (3.10) for any \( \rho > 0 \). Then (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.11).

Now we can prove the main result of the section.

**Theorem 3.4.** Let us suppose that assumptions \((g'^1) - (g'^3)\) hold and let \( V \in \mathcal{V}(\Omega) \). Then, for every \( \lambda \in \mathbb{R} \), problem (3.1) has a nontrivial solution \( u \in W_0^{1,p}(\Omega) \) in each of the following cases:

\( (a) \) there exists \( \delta > 0 \) such that \( G(x, s) \geq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \),

\( (b) \) there exists \( \delta > 0 \) such that \( G(x, s) \leq 0 \) for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \) with \( |s| \leq \delta \).

**Proof.** A standard argument shows that \( \Phi \) satisfies the Palais-Smale compactness condition (see, e.g., [9, Proposition 4.3]).

Suppose, for a contradiction, that the origin is the only critical point of \( \Phi \). By Lemma 3.2 there exists \( a < 0 \) such that \( \Phi^a \) is contractible in itself. On the other hand, by the second deformation lemma (see e.g. Chang [3] or Mawhin and Willem [17]), \( \Phi^0 \) is a deformation retract of \( W \) and \( \Phi^a \) is a deformation retract of \( \Phi^0 \setminus \{0\} \), so

\[
C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}) \approx H^q(W, \Phi^0) = 0 \quad \text{for every } q \geq 0.
\]

By Lemma 3.3 a contradiction follows.
For the sake of completeness, let us state a simple extension of a result of Perera [19], which can be proved by the same argument.

**Theorem 3.5.** Let us suppose that assumptions $(g1') - (g3')$ hold, let $V \in \mathcal{V}(\Omega)$ and let $\lambda \geq 0$. If the set $\{ x \in \Omega : V(x) > 0 \}$ has positive measure, assume also that $\lambda \notin \{ \lambda_k : k \geq 1 \}$, where $(\lambda_k)$ is the sequence defined in (2.4).

Then problem (3.1) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$.

Thus, the extra assumption on $\lambda$ is compensated by the fact that there is no sign condition on $G$. Observe that the union of Theorems 3.4 and 3.5 provides a complete extension to the case $p \neq 2$ of S.J. Li and Willem [15, Theorem 4].

### 4 Proof of Theorem 1.1

Since $V \in L^\infty(\Omega)$, we have $V \in \mathcal{V}(\Omega)$. It is also standard that assumptions $(g1) - (g3)$ imply $(g1') - (g3')$ (see e.g. Degiovanni and Lancelotti [9]). Then the assertion follows from Theorem 3.4.

### References


