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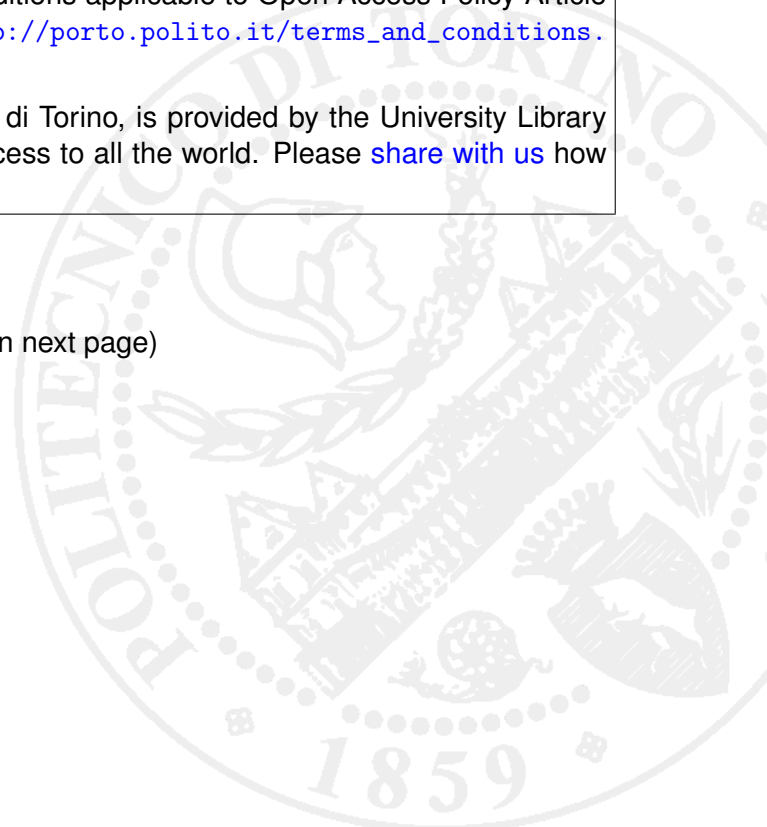
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Nontrivial Solutions of p -Superlinear p -Laplacian Problems via a Cohomological Local Splitting

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Abstract

We consider a quasilinear equation, involving the p -Laplace operator, with a p -superlinear nonlinearity. We prove the existence of a nontrivial solution, also when there is no mountain pass geometry, without imposing a global sign condition. Techniques of Morse theory are employed.

Keywords: p -Laplace equations; nontrivial solutions; Morse theory.

Mathematics Subject Classification 2010: 58E05, 35J65

1 Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda V |u|^{p-2} u + g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded open subset of \mathbb{R}^n , $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian of u , $p \in]1, \infty[$, $\lambda \in \mathbb{R}$ is a parameter, $V \in L^\infty(\Omega)$ and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the following conditions:

(g1) there exist $C > 0$ and

$$p < q < \begin{cases} p^* := \frac{np}{n-p} & \text{if } p < n \\ \infty & \text{if } p \geq n \end{cases}$$

such that

$$|g(x, s)| \leq C (|s|^{q-1} + 1) ;$$

(g2) we have

$$g(x, s) = o(|s|^{p-1}) \text{ as } s \rightarrow 0, \text{ uniformly in } x ;$$

(g3) there exist $\mu > p$ and $R > 0$ such that

$$0 < \mu G(x, s) := \int_0^s g(x, t) dt \leq s g(x, s), \quad \text{whenever } |s| \geq R.$$

In particular, $g(x, 0) \equiv 0$ and hence we have the trivial solution $u = 0$, and we seek another.

In the case $p = 2$, the existence of a nontrivial solution u for (1.1) can be obtained via the Linking Theorem (see e.g. Rabinowitz [21, Theorem 5.16]). More precisely, let us assume, without loss of generality, that $\lambda \geq 0$. If the set

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} V |u|^p dx = 1 \right\}$$

is empty or if $\mathcal{M} \neq \emptyset$ and

$$\lambda < \lambda_1 := \min \left\{ \int_{\Omega} |\nabla u|^p dx : u \in \mathcal{M} \right\},$$

then the existence of a nontrivial solution can be proved, without any further assumption, by the Mountain Pass Theorem for any $p > 1$ (see Ambrosetti and Rabinowitz [1] for the case $p = 2$ and Dinca, Jebelean and Mawhin [10] for the case $p \neq 2$). On the contrary, if $\mathcal{M} \neq \emptyset$ and $\lambda \geq \lambda_1$, the classical proof is based on the fact that each eigenvalue λ_k of $-\Delta_2$ induces a suitable direct sum decomposition of $W_0^{1,2}(\Omega)$. On the other hand, if $p \neq 2$, such decompositions are not available. Nevertheless, a linking argument over cones, rather than over linear subspaces, has been developed for $p \neq 2$, when λ is close to λ_1 by Fan and Z. Li [12] and for any λ by Degiovanni and Lancelotti [9]. In such a way, the mentioned result of Rabinowitz has been completely extended to the case $p \neq 2$.

When $\lambda \geq \lambda_1$, in all these results a global sign condition like $G(x, s) \geq 0$ needs to be imposed, in order to recognize the linking geometry. However, such an assumption can be relaxed by means of Morse theory or nonstandard linking constructions.

When $p = 2$, in Benci [2, Theorem 7.14] it is shown that, if a nonresonance condition at the origin is satisfied, the existence of a nontrivial solution can be obtained without any further assumption. On the other hand, S.J. Li and Willem [15, Theorem 4] are able to treat the resonant case under a local sign condition on G . Related results are also contained in J.Q. Liu and S.J. Li [14].

The approach based on Morse theory has been extended to the case $p \neq 2$ by S. Liu [16] when λ is close to λ_1 and by Perera [19] when λ does not belong to the spectrum of the p -Laplace operator.

Our purpose is to develop this approach, in order to remove any condition on λ and require only a local sign condition on G . Our result is the following

Theorem 1.1. *Let us suppose that assumptions (g1) – (g3) hold and let $V \in L^\infty(\Omega)$. Then, for every $\lambda \in \mathbb{R}$, problem (1.1) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$ in each of the following cases:*

- (a) *there exists $\delta > 0$ such that $G(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$;*
- (b) *there exists $\delta > 0$ such that $G(x, s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$.*

This is a natural extension to the case $p \neq 2$ of the mentioned result of S.J. Li and Willem, although the argument is based there on a nonstandard linking construction and here on Morse theory.

In the next section we recall and prove some preliminary facts, while in section 3 we prove the main result in a more general setting. In the last section we recover Theorem 1.1 as a particular case.

2 Preliminaries

Let Φ be a C^1 -functional defined on a real Banach space W . We denote by B_ρ and S_ρ the closed ball and sphere of center 0 and radius ρ . We also denote by H the Alexander-Spanier cohomology with \mathbb{Z}_2 -coefficients (see Spanier [22]). For a symmetric subset X of $W \setminus \{0\}$, $i(X)$ denotes its \mathbb{Z}_2 -cohomological index (see Fadell and Rabinowitz [11]). The following notion has been introduced, in a slightly different form, by Perera, Agarwal, and O'Regan [20] and is in turn a variant of the homological local linking of Perera [18]. It should also be compared with the local linking of S.J. Li and Willem [15].

Definition 2.1. We say that Φ has a cohomological local splitting near 0 in dimension $k < \infty$, if there are two symmetric cones W_-, W_+ in W and $\rho > 0$ such that

$$W_- \cap W_+ = \{0\}, \quad i(W_- \setminus \{0\}) = i(W \setminus W_+) = k \quad (2.1)$$

and

$$\begin{cases} \Phi(u) \leq \Phi(0) & \text{for every } u \in B_\rho \cap W_-, \\ \Phi(u) \geq \Phi(0) & \text{for every } u \in B_\rho \cap W_+. \end{cases} \quad (2.2)$$

As we will see, in such a case 0 must be a critical point of Φ .

Recall that the cohomological critical groups of Φ at a point $u \in W$ are defined by

$$C^q(\Phi, u) = H^q(\Phi^c, \Phi^c \setminus \{u\}), \quad q \geq 0,$$

where $c = \Phi(u)$ is the corresponding value and Φ^c is the closed sublevel set $\{w \in W : \Phi(w) \leq c\}$ (see, e.g., Chang [3] or Mawhin and Willem [17]). By the excision property, we have

$$C^q(\Phi, u) \approx H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\})$$

for every neighborhood U of u . Therefore, the concept has local nature. Moreover, it is well known that all critical groups are trivial, if u is not a critical point of Φ (see e.g. Corvellec [5, Proposition 3.4]). Finally, the next result shows a stability property and is a particular case of Corvellec and Hantoute [7, Theorem 5.2] (see also Benci [2, Theorem 5.16]). The case in which W is a Hilbert space and Φ is of class C^2 can be also found in Chang [3, Theorem I.5.6] and in Mawhin and Willem [17, Theorem 8.8].

Theorem 2.2. *Let $\Phi_t : W \rightarrow \mathbb{R}$, $t \in [0, 1]$, be a family of functionals of class C^1 . Assume that there exists $\rho > 0$ such that each Φ_t satisfies the Palais-Smale condition over B_ρ and has no critical point in B_ρ other than 0. Suppose also that the map $\{t \mapsto \Phi_t\}$ is continuous from $[0, 1]$ into $C^1(B_\rho)$.*

Then $C^q(\Phi_t, 0)$ is independent of t .

The cohomological local splitting allows to give an estimate of the critical groups, also in the absence of a direct sum decomposition.

Proposition 2.3. *If Φ has a cohomological local splitting near 0 in dimension k , then 0 is a critical point of Φ . Moreover, if 0 is an isolated critical point of Φ , then we have $C^k(\Phi, 0) \neq 0$.*

This proposition is a variant of a result of Perera, Agarwal, and O'Regan [20]. We need the following lemma from Degiovanni and Lancelotti (see [9, Theorem 2.7] and also Cingolani and Degiovanni [4, Theorem 3.6]), which establishes a connection between equivariant index and nonequivariant cohomology.

Lemma 2.4. *If X is a symmetric subset of $W \setminus \{0\}$ with $k = i(X) < \infty$ and A is a symmetric subset of X with $i(A) = k$, then the homomorphism $i^* : H^k(W, X) \rightarrow H^k(W, A)$, induced by the inclusion $i : (W, A) \subseteq (W, X)$, is nontrivial.*

Proof of Proposition 2.3. It is enough to prove that, if 0 does not accumulate critical points of Φ , then $C^k(\Phi, 0) \neq 0$. Therefore assume, without loss of generality, that Φ has no critical point u with $0 < \|u\| \leq \rho$. Let $c = \Phi(0)$.

There exists a deformation $\eta : W \times [0, 1] \rightarrow W$ such that

$$\begin{aligned} \Phi(\eta(u, t)) &< \Phi(u) && \text{if } \Phi'(u) \neq 0 \text{ and } t > 0, \\ \eta(u, t) &= u && \text{otherwise,} \end{aligned}$$

(see e.g. Benci [2, Theorem 5.5] or Corvellec [6]). Let $0 < r \leq \rho$ be such that $\eta(B_r \times [0, r]) \subseteq B_\rho$. Since $B_r \cap W_-$ is contractible and $S_r \cap W_-$ is a deformation retract of $W_- \setminus \{0\}$, from Lemma 2.4 and (2.1) we deduce that the homomorphism

$$i^* : H^k(W, B_\rho \setminus W_+) \longrightarrow H^k(B_r \cap W_-, S_r \cap W_-),$$

induced by the inclusion $i : (B_r \cap W_-, S_r \cap W_-) \subseteq (W, B_\rho \setminus W_+)$, is nontrivial. On the other hand, since (2.2) implies

$$B_r \cap W_- \subseteq \Phi^c \cap B_r, \quad S_r \cap W_- \subseteq \Phi^c \cap B_r \setminus \{0\}, \quad \eta(\Phi^c \cap B_r \setminus \{0\}, r) \subseteq B_\rho \setminus W_+,$$

we may also consider the composition

$$H^k(W, B_\rho \setminus W_+) \xrightarrow{\eta(\cdot, r)^*} H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \xrightarrow{j^*} H^k(B_r \cap W_-, S_r \cap W_-)$$

where $j : (B_r \cap W_-, S_r \cap W_-) \subseteq (\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\})$ is the inclusion. Again (2.2) yields

$$\eta((S_r \cap W_-) \times [0, r]) \subseteq B_\rho \setminus W_+,$$

so that $\eta(\cdot, r) \circ j$ is homotopic to i . Therefore $j^* \circ \eta(\cdot, r)^* = i^*$ is nontrivial, which in turn implies that $H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \neq 0$. \square

Now, let us recall a situation in which one can build two symmetric cones satisfying (2.1). Let Ω be a bounded open subset of \mathbb{R}^n , let $1 < p < \infty$ and let

$$\mathcal{V}(\Omega) := \begin{cases} \bigcup_{r>n/p} L^r(\Omega) & \text{if } p \leq n, \\ L^1(\Omega) & \text{if } p > n. \end{cases}$$

Take $V \in \mathcal{V}(\Omega)$ and consider the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda V |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

We refer the reader to Cuesta [8] and Szulkin and Willem [23] for general properties concerning (2.3).

Now, assume that $\{x \in \Omega : V(x) > 0\}$ has positive measure, denote by \mathcal{F} the class of symmetric subsets of

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} V |u|^p dx = 1 \right\}$$

and set

$$\lambda_k = \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \int_{\Omega} |\nabla u|^p dx, \quad k \geq 1. \quad (2.4)$$

Then $\lambda_k \nearrow +\infty$ are eigenvalues of (2.3) and the following result holds (see Degiovanni and Lancelotti [9, Theorem 3.2]).

Proposition 2.5. *Let $k \geq 1$ be such that $\lambda_k < \lambda_{k+1}$ and let*

$$\begin{aligned} W_- &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p dx \leq \lambda_k \int_{\Omega} V |u|^p dx \right\}, \\ W_+ &= \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p dx \geq \lambda_{k+1} \int_{\Omega} V |u|^p dx \right\}. \end{aligned}$$

Then W_-, W_+ are two symmetric cones in $W_0^{1,p}(\Omega)$ satisfying (2.1).

3 The main result

Let Ω be a bounded open subset of \mathbb{R}^n , let $1 < p < \infty$, let $V \in \mathcal{V}(\Omega)$ and let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the following assumptions:

(g1') we have that

for every $\varepsilon > 0$ there exists $a_\varepsilon \in \mathcal{V}(\Omega)$ such that

$$|g(x, s)| \leq a_\varepsilon(x) |s|^{p-1} + \varepsilon |s|^{p^*-1}, \quad \text{if } p < n;$$

there exist $a \in \mathcal{V}(\Omega)$, $C > 0$ and $q > p$ such that

$$|g(x, s)| \leq a(x) |s|^{p-1} + C |s|^{q-1}, \quad \text{if } p = n;$$

for every $S > 0$ there exists $a_S \in \mathcal{V}(\Omega)$ such that

$$|g(x, s)| \leq a_S(x) |s|^{p-1} \text{ whenever } |s| \leq S, \quad \text{if } p > n;$$

(g2') for a.e. $x \in \Omega$, we have $\lim_{s \rightarrow 0} \frac{G(x, s)}{|s|^p} = 0$ and $\lim_{|s| \rightarrow \infty} \frac{G(x, s)}{|s|^p} = +\infty$, where

$$G(x, s) = \int_0^s g(x, t) dt;$$

(g3') there exist $\mu > p$, $\gamma_0 \in L^1(\Omega)$ and $\gamma_1 \in \mathcal{V}(\Omega)$ such that

$$-\gamma_0(x) - \gamma_1(x) |s|^p \leq \mu G(x, s) \leq s g(x, s) + \gamma_0(x) + \gamma_1(x) |s|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

In order to study the quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda V |u|^{p-2} u + g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

let us define a functional $\Phi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ of class C^1 by

$$\Phi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{p} \int_\Omega V |u|^p dx - \int_\Omega G(x, u) dx$$

and set $\|u\| = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}$ for every $u \in W_0^{1,p}(\Omega)$. Recall also that, for every $\gamma \in \mathcal{V}(\Omega)$, the map $\{u \mapsto \gamma |u|^p\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ into $L^1(\Omega)$.

Lemma 3.1. *The following facts hold:*

(a) *for every $c \in \mathbb{R}$, we have*

$$\limsup_{\substack{\|u\| \rightarrow \infty \\ \Phi(u) \leq c}} \frac{\Phi'(u)u - p\Phi(u)}{\|u\|^p} < 0;$$

(b) *for every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have*

$$\lim_{|t| \rightarrow \infty} \frac{\Phi(tu)}{|t|^p} = -\infty.$$

Proof. (a) Let $c \in \mathbb{R}$. By contradiction, let $d_k \rightarrow 0$ and let (u_k) be a sequence in Φ^c such that $\|u_k\| \rightarrow \infty$ and

$$\Phi'(u_k)u_k - p\Phi(u_k) \geq -d_k\|u_k\|^p \quad \text{for every } k \in \mathbb{N}.$$

If we set $v_k = u_k/\|u_k\|$, up to a subsequence (v_k) is convergent to some $v \in W_0^{1,p}(\Omega)$ weakly and a.e. in Ω .

From $(g3')$ it follows that

$$\begin{aligned} -d_k\|u_k\|^p &\leq \Phi'(u_k)u_k - p\Phi(u_k) = \int_{\Omega} (pG(x, u_k) - u_k g(x, u_k)) \, dx \\ &= \int_{\Omega} (\mu G(x, u_k) - u_k g(x, u_k)) \, dx - (\mu - p) \int_{\Omega} G(x, u_k) \, dx \\ &\leq \int_{\Omega} (\gamma_0 + \gamma_1|u_k|^p) \, dx - (\mu - p) \int_{\Omega} G(x, u_k) \, dx, \end{aligned}$$

whence

$$(\mu - p) \int_{\Omega} G(x, u_k) \, dx \leq d_k\|u_k\|^p + \int_{\Omega} (\gamma_0 + \gamma_1|u_k|^p) \, dx. \quad (3.2)$$

Therefore we have

$$\limsup_k \frac{\int_{\Omega} G(x, u_k) \, dx}{\|u_k\|^p} < +\infty. \quad (3.3)$$

On the other hand, $(g3')$ also yields

$$\frac{G(x, u_k)}{\|u_k\|^p} + \frac{\gamma_0}{\|u_k\|^p} \geq -\gamma_1|v_k|^p,$$

hence, by the (generalized) Fatou lemma and (3.3),

$$\int_{\Omega} \left(\liminf_k \frac{G(x, u_k)}{\|u_k\|^p} \right) dx \leq \liminf_k \frac{\int_{\Omega} G(x, u_k) dx}{\|u_k\|^p} < +\infty.$$

Since by (g2') we have

$$\lim_k \frac{G(x, u_k)}{\|u_k\|^p} = \lim_k \left(\frac{G(x, u_k)}{|u_k|^p} |v_k|^p \right) = +\infty \quad \text{where } v \neq 0,$$

we deduce that $v = 0$ a.e. in Ω .

Formula (3.2) can also be rewritten as

$$\begin{aligned} \left(\frac{\mu}{p} - 1 \right) \int_{\Omega} |\nabla u_k|^p dx &\leq (\mu - p)\Phi(u_k) + d_k \|u_k\|^p \\ &\quad + \int_{\Omega} \gamma_0 dx + \int_{\Omega} \left[\left(\frac{\mu}{p} - 1 \right) V + \gamma_1 \right] |u_k|^p dx, \end{aligned}$$

namely

$$\left(\frac{\mu}{p} - 1 \right) \leq d_k + (\mu - p) \frac{\Phi(u_k)}{\|u_k\|^p} + \frac{\int_{\Omega} \gamma_0 dx}{\|u_k\|^p} + \int_{\Omega} \left[\left(\frac{\mu}{p} - 1 \right) V + \gamma_1 \right] |v_k|^p dx.$$

Going to the limit as $k \rightarrow \infty$, we get $\frac{\mu}{p} - 1 \leq 0$ and a contradiction follows.

(b) Since by (g3')

$$\frac{G(x, tu)}{|t|^p} + \frac{\gamma_0}{|t|^p} \geq -\gamma_1 |u|^p,$$

applying as before Fatou's lemma, the assertion follows. \square

Lemma 3.2. *There exists $a < 0$ such that Φ^a is contractible in itself.*

Proof. By (a) of Lemma 3.1, there exists $b \in \mathbb{R}$ such that

$$\Phi'(u)u - p\Phi(u) \leq b \quad \text{for every } u \in \Phi^0.$$

In particular, there exists $a < 0$ such that

$$\Phi'(u)u < 0 \quad \text{for every } u \in \Phi^a. \quad (3.4)$$

If we set, taking into account (b) of Lemma 3.1,

$$t(u) = \min \{t \geq 1 : \Phi(tu) \leq a\},$$

from (3.4) we deduce that the function $\{u \mapsto t(u)\}$ is continuous. Then $r(u) = t(u)u$ is a retraction of $W_0^{1,p}(\Omega) \setminus \{0\}$ onto Φ^a . Since $W_0^{1,p}(\Omega) \setminus \{0\}$ is contractible in itself, the same is true for Φ^a . \square

Lemma 3.3. *Assume that 0 is an isolated critical point of Φ . Then the following facts hold:*

- (a) *if there exists $\delta > 0$ such that $G(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$, we have $C^q(\Phi, 0) \neq 0$ for*

$$q = i \left(\left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : \int_{\Omega} |\nabla u|^p dx \leq \lambda \int_{\Omega} V |u|^p dx \right\} \right);$$

- (b) *if there exists $\delta > 0$ such that $G(x, s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$, we have $C^q(\Phi, 0) \neq 0$ for*

$$q = i \left(\left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p dx < \lambda \int_{\Omega} V |u|^p dx \right\} \right).$$

Proof. By replacing (λ, V) with $(-\lambda, -V)$ if necessary, we may assume that $\lambda \geq 0$. Let $\vartheta : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ -function such that $\vartheta(s) = 0$ for $|s| \leq \delta/2$ and $\vartheta(s) = 1$ for $|s| \geq \delta$. For every $t \in [0, 1]$, define

$$G_t(x, s) = G(x, (1 - t\vartheta(s))s)$$

and $\Phi_t : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by

$$\Phi_t(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\Omega} V |u|^p dx - \int_{\Omega} G_t(x, u) dx.$$

From $(g1')$ it follows that each Φ_t satisfies the Palais-Smale condition over every bounded subset of $W_0^{1,p}(\Omega)$ (see also [9, Proposition 4.3]). Moreover the map $\{t \mapsto \Phi_t\}$ is continuous from $[0, 1]$ into $C^1(B)$ for every bounded subset B of $W_0^{1,p}(\Omega)$. We claim that there exists $\rho > 0$ such that each Φ_t has no critical point in B_ρ other than 0. By contradiction, let (t_j) be a sequence in $[0, 1]$ and (u_j) a sequence convergent to 0 with $\Phi'_{t_j}(u_j) = 0$

and $u_j \neq 0$. Then the same regularity argument of Guedda and Veron [13, Propositions 1.2 and 1.3] shows that (u_j) is convergent to 0 also in $L^\infty(\Omega)$. Therefore we have $\Phi'(u_j) = 0$ eventually as $j \rightarrow \infty$. Since 0 is an isolated critical point of Φ , a contradiction follows. From Theorem 2.2 we deduce that $C^q(\Phi, 0) \approx C^q(\Phi_1, 0)$ for any $q \geq 0$.

Observe also that

$$\int_{\Omega} G_1(x, u) dx = o(\|u\|^p) \text{ as } \|u\| \rightarrow 0 \quad (3.5)$$

(see, e.g., [9, Proposition 4.3]).

In case (a), we have $G_1(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. If the set $\{x \in \Omega : V(x) > 0\}$ has positive measure and $\lambda \geq \lambda_1$, where (λ_k) is the sequence defined in (2.4), take $k \geq 1$ such that $\lambda_k \leq \lambda < \lambda_{k+1}$ and define W_-, W_+ as in Proposition 2.5. Otherwise, let $W_- = \{0\}$ and $W_+ = W_0^{1,p}(\Omega)$. In any case, W_-, W_+ are two symmetric cones in $W_0^{1,p}(\Omega)$ satisfying (2.1) and

$$\begin{aligned} & i(W_- \setminus \{0\}) \\ &= i \left(\left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : \int_{\Omega} |\nabla u|^p dx \leq \lambda \int_{\Omega} V |u|^p dx \right\} \right), \end{aligned} \quad (3.6)$$

$$\int_{\Omega} |\nabla u|^p dx \leq \lambda \int_{\Omega} V |u|^p dx \quad \forall u \in W_-, \quad (3.7)$$

$$\exists \sigma \in]0, 1[: (1 - \sigma) \int_{\Omega} |\nabla u|^p dx \geq \lambda \int_{\Omega} V |u|^p dx \quad \forall u \in W_+. \quad (3.8)$$

From (3.7) and the sign information on G_1 , it follows

$$\Phi_1(u) \leq 0 \quad \text{for every } u \in W_- \text{ with } \|u\| \leq \rho \quad (3.9)$$

for any $\rho > 0$. On the other hand, combining (3.5) and (3.8), we get

$$\Phi_1(u) \geq 0 \quad \text{for every } u \in W_+ \text{ with } \|u\| \leq \rho \quad (3.10)$$

provided that ρ is sufficiently small. Therefore also (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.6).

In case (b), we have $G_1(x, s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. If the set $\{x \in \Omega : V(x) > 0\}$ has positive measure and $\lambda > \lambda_1$, take now

$k \geq 1$ such that $\lambda_k < \lambda \leq \lambda_{k+1}$ and define W_-, W_+ as in Proposition 2.5. Otherwise, let $W_- = \{0\}$ and $W_+ = W_0^{1,p}(\Omega)$. In any case, W_-, W_+ are two symmetric cones in $W_0^{1,p}(\Omega)$ satisfying (2.1) and

$$\begin{aligned} & i(W_0^{1,p}(\Omega) \setminus W_+) \\ &= i\left(\left\{u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p dx < \lambda \int_{\Omega} V|u|^p dx\right\}\right), \end{aligned} \quad (3.11)$$

$$\exists \sigma \in]0, 1[: (1 + \sigma) \int_{\Omega} |\nabla u|^p dx \leq \lambda \int_{\Omega} V|u|^p dx \quad \forall u \in W_-, \quad (3.12)$$

$$\int_{\Omega} |\nabla u|^p dx \geq \lambda \int_{\Omega} V|u|^p dx \quad \forall u \in W_+. \quad (3.13)$$

Combining (3.5) with (3.12), we get again (3.9) if ρ is sufficiently small. On the other hand, from (3.13) and the sign information on G_1 we deduce (3.10) for any $\rho > 0$. Then (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.11). \square

Now we can prove the main result of the section.

Theorem 3.4. *Let us suppose that assumptions $(g1')$ – $(g3')$ hold and let $V \in \mathcal{V}(\Omega)$. Then, for every $\lambda \in \mathbb{R}$, problem (3.1) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$ in each of the following cases:*

- (a) *there exists $\delta > 0$ such that $G(x, s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$,*
- (b) *there exists $\delta > 0$ such that $G(x, s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$.*

Proof. A standard argument shows that Φ satisfies the Palais-Smale compactness condition (see, e.g., [9, Proposition 4.3]).

Suppose, for a contradiction, that the origin is the only critical point of Φ . By Lemma 3.2 there exists $a < 0$ such that Φ^a is contractible in itself. On the other hand, by the second deformation lemma (see e.g. Chang [3] or Mawhin and Willem [17]), Φ^0 is a deformation retract of W and Φ^a is a deformation retract of $\Phi^0 \setminus \{0\}$, so

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}) \approx H^q(W, \Phi^a) = 0 \quad \text{for every } q \geq 0.$$

By Lemma 3.3 a contradiction follows. \square

For the sake of completeness, let us state a simple extension of a result of Perera [19], which can be proved by the same argument.

Theorem 3.5. *Let us suppose that assumptions $(g1')$ – $(g3')$ hold, let $V \in \mathcal{V}(\Omega)$ and let $\lambda \geq 0$. If the set $\{x \in \Omega : V(x) > 0\}$ has positive measure, assume also that $\lambda \notin \{\lambda_k : k \geq 1\}$, where (λ_k) is the sequence defined in (2.4).*

Then problem (3.1) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$.

Thus, the extra assumption on λ is compensated by the fact that there is no sign condition on G . Observe that the union of Theorems 3.4 and 3.5 provides a complete extension to the case $p \neq 2$ of S.J. Li and Willem [15, Theorem 4].

4 Proof of Theorem 1.1

Since $V \in L^\infty(\Omega)$, we have $V \in \mathcal{V}(\Omega)$. It is also standard that assumptions $(g1) - (g3)$ imply $(g1') - (g3')$ (see e.g. Degiovanni and Lancelotti [9]). Then the assertion follows from Theorem 3.4.

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