Analysis of laminated shells by a sinusoidal shear deformation theory and radial basis functions collocation, accounting for through-the-thickness deformations

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A B S T R A C T

In this paper, the static and free vibration analysis of laminated shells is performed by radial basis functions collocation, according to a sinusoidal shear deformation theory (SSDT). The SSDT theory accounts for through-the-thickness deformation, by considering a sinusoidal evolution of all displacements with the thickness coordinate. The equations of motion and the boundary conditions are obtained by the Carrera’s Unified Formulation, and further interpolated by collocation with radial basis functions.

1. Introduction

The efficient load-carrying capabilities of shell structures make them very useful in a variety of engineering applications. The continuous development of new structural materials leads to ever increasingly complex structural designs that require careful analysis. Although analytical techniques are very important, the use of numerical methods to solve shell mathematical models of complex structures has become an essential ingredient in the design process.

The most common mathematical models used to describe shell structures may be classified in two classes according to different physical assumptions: the Koiter model [1], based on the Kirchhoff hypothesis and the Naghdi model [2], based on the Reissner–Mindlin assumptions that take into account the transverse shear deformation. In this paper, the Unified Formulation (UF) by Carrera [3–8] is proposed to derive the equations of motion and boundary conditions. This formulation contains a large variety of 2D models that differ in the order of used expansion in thickness direction and in the manner the variables are modelled along the thickness. In particular, the UF is here applied to analyse laminated shells by radial basis functions (RBF) collocation, according to a sinus-based shear deformation theory that accounts for through-the-thickness deformations. This theory is an expansion of the developments by Touratier [9–11], and Vidal and Polit [12]. It allows a sinusoidal variation of all displacement components along the thickness and it is more convenient than the classical Taylor polynomials because the sine function can be expressed by means of Taylor expansion. Moreover, the derivative of displacements in the deformations does not reduce the order of the approximating functions. The choice of the sine function can be justified from the three-dimensional point of view, using the work of Cheng [13]. As it can be seen in Polit [14], a sine term appears in the solution of the shear equation (see Eq. (7) in [2]). Therefore, the kinematics proposed can be seen as an approximation of the exact three-dimensional solution. Furthermore, the sine function has an infinite radius of convergence and its Taylor expansion includes not only the third-order terms but all the odd terms.

The most common numerical procedure for the analysis of the shells is the finite element method [15–19]. It is known that the phenomenon of numerical locking may arise from hidden constraints that are not well represented in the finite element approximation and, in scientific literature, it is possible to find many methods to overcome this problem [20–25]. The present paper, that performs the bending and free vibration analysis of laminated shells by collocation with radial basis functions, avoids the locking phenomenon. A radial basis function, \( \phi(x) \) is a spline that depends on the Euclidian distance between distinct data centers \( x_{ij} = 1,2,\ldots,N \in \mathbb{R}^n \), also called nodal or collocation points. The
A sinus shear deformation theory

The present sinus shear deformation theory involves the following expansion of displacements

\[
\begin{align*}
  u &= u_0 + Z u_1 + \sin \left( \frac{\pi z}{h} \right) u_3; \\
  v &= v_0 + Z v_1 + \sin \left( \frac{\pi z}{h} \right) v_3; \\
  w &= w_0 + Z w_1 + \sin \left( \frac{\pi z}{h} \right) w_3
\end{align*}
\]

where \(u_0, v_0, w_0\) are translations of a point at the middle-surface of the plate, and \(u_1, v_1, u_3, v_3\) denote rotations. This theory is an expansion of early developments by Touratier [9–11], and Vidal and Polit [12]. It considers a sinusoidal variation of all displacements \(u, v, w\), allowing for through-the-thickness deformations. Extension to shells becomes evident in the next sections.

3.1. Governing equations and boundary conditions in the framework of Unified Formulation

Shells are bi-dimensional structures in which one dimension (in general the thickness in \(z\) direction) is negligible with respect to the other two in-plane dimensions. Geometry and the reference system are indicated in Fig. 1. The square of an infinitesimal linear segment in the layer, the associated infinitesimal area and volume are given by:

\[
\begin{align*}
  ds_k^2 &= H_k^{11} dx^2 + H_k^{22} dy^2 + H_k^{33} dz^2 \\
  d\Omega_k &= H_k^{11} H_k^{22} dx dy \\
  dV &= H_k^{11} H_k^{22} H_k^{33} dx dy dz
\end{align*}
\]

where the metric coefficients are:

\[
\begin{align*}
  H_k^{ij} &= \frac{1}{2} \left( \frac{\partial x_i}{\partial n} \frac{\partial x_j}{\partial n} + \frac{\partial x_i}{\partial n} \frac{\partial x_j}{\partial n} + \frac{\partial x_i}{\partial \tilde{n}} \frac{\partial x_j}{\partial \tilde{n}} \right)
\end{align*}
\]

\(k\) denotes the \(k\)-layer of the multilayered shell; \(R_k^1\) and \(R_k^2\) are the principal radii of curvature along the coordinates \(\tilde{x}\) and \(\tilde{y}\), respectively. \(A^k\) and \(B^k\) are the coefficients of the first fundamental form of \(\Omega_k\) (\(\Omega_k\) is the \(\Omega_k\) boundary). In this work, the attention has been restricted to shells with constant radii of curvature (cylindrical, spherical, toroidal geometries) for which \(A^k = B^k = 1\).

Although one can use the UF for one-layer, isotropic shell, a multi-layered shell with \(N_l\) layers is considered. The Principle of Virtual Displacements (PVD) for the pure-mechanical case reads:

\[
\sum_{k=1}^{N_l} \int_{\Omega_k} \left\{ \delta \varepsilon_{pc}^k \sigma_{pc}^k + \delta \varepsilon_{nc}^k \sigma_{nc}^k \right\} d\Omega_k d\zeta = \sum_{k=1}^{N_l} \delta \Omega_k
\]

where \(\Omega_k\) and \(A_k\) are the integration domains in plane \((x, y)\) and \(z\) direction, respectively. Here, \(k\) indicates the layer and \(T\) the transpose of a vector, and \(\delta \Omega_k\) is the external work for the \(k\)th layer. \(G\) means geometrical relations and \(C\) constitutive equations.

The steps to obtain the governing equations are:

- Substitution of the geometrical relations (subscript \(G\)).
- Substitution of the appropriate constitutive equations (subscript \(C\)).
- Introduction of the Unified Formulation.

Stresses and strains are separated into in-plane and normal components, denoted, respectively, by the subscripts \(p\) and \(n\). The mechanical strains in the \(k\)th layer can be related to the displacement field \(\mathbf{u}^k = \{u^k, v^k, w^k\}^T\) via the geometrical relations:

\[
\begin{align*}
  \varepsilon_{pc}^k &= \begin{bmatrix} \varepsilon_{xx}^k & \varepsilon_{xy}^k & \varepsilon_{xz}^k \end{bmatrix}^T = (\mathbf{D}_p^k A_k) \mathbf{u}^k, \\
  \varepsilon_{nc}^k &= \begin{bmatrix} \varepsilon_{zz}^k & \varepsilon_{zy}^k & \varepsilon_{zy}^k \end{bmatrix}^T = (\mathbf{D}_n^k A_k) \mathbf{u}^k
\end{align*}
\]

The explicit form of the introduced arrays follows:

\[
\mathbf{D}_p^k = \begin{bmatrix} D_{11}^k & 0 & 0 \\ 0 & D_{22}^k & 0 \\ 0 & 0 & D_{33}^k \end{bmatrix}, \quad \mathbf{D}_n^k = \begin{bmatrix} 0 & 0 & \varepsilon_{zz}^k \\ 0 & 0 & \varepsilon_{zy}^k \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{\Omega_k}^k = \begin{bmatrix} \partial z & 0 & 0 \\ 0 & \partial z & 0 \\ 0 & 0 & \partial z \end{bmatrix}
\]
The 3D constitutive equations are given as:

\[
\sigma_{pc} = C_{pp}^{c} \epsilon_{pc} + C_{pm}^{c} \epsilon_{mc} \\
\sigma_{ac} = C_{am}^{c} \epsilon_{pc} + C_{am}^{c} \epsilon_{ac}
\]

with

\[
C_{pp}^{c} = \begin{bmatrix}
C_{11} & C_{12} & C_{14} \\
C_{12} & C_{22} & C_{24} \\
C_{14} & C_{24} & C_{66}
\end{bmatrix}, \\
C_{pm}^{c} = \begin{bmatrix}
0 & 0 & C_{13} \\
0 & 0 & C_{23} \\
0 & 0 & C_{33}
\end{bmatrix}, \\
C_{am}^{c} = \begin{bmatrix}
C_{45} & C_{55} & 0 \\
C_{45} & C_{55} & 0 \\
0 & 0 & C_{55}
\end{bmatrix}
\]

According to the Unified Formulation by Carrera, the three displacement components \(u_x, u_y\) and \(u_z\) and their relative variations can be modelled as:

\[
\begin{align*}
\langle u_x, u_y, u_z \rangle &= F_i (u_x, u_y, u_z), \\
\frac{\partial u_x}{\partial x} &= \gamma_{x} (\text{dis}), \frac{\partial u_y}{\partial y} = \gamma_{y} (\text{dis}), \frac{\partial u_z}{\partial z} = \gamma_{z} (\text{dis}),
\end{align*}
\]

where the \(F_i\) are functions of the thickness coordinate \(z\) and \(x, y, z\) is a sum index. Taylor expansions from first- to fourth-order are employed: \(F_0 = z^0 = 1, F_1 = z^1 = z, \ldots, F_8 = z^8, \ldots, F_8 = z^8\) if an Equivalent Single Layer (ESL) model is used. For ESL approach, one means that the displacements are assumed for the whole laminate if a multi-fiber strategy is considered.

Resorting to the displacement field in Eq. (10), the vectors \(F_i\) are chosen for the displacements \(u, v, w\). Then, all the terms of the equations of motion are obtained by integrating through the thickness direction.

It is interesting to note that under this combination of the Unified Formulation and RBF collocation, the collocation code depends only on the choice of \(F_i\), in order to solve this type of problems. A MATLAB code has been designed that, just by changing \(F_i\), can analyse static deformations, and free vibrations for any type of \(C^s\) shear deformation theory. An obvious advantage of the present methodology is that the tedious derivation of the equations of motion and boundary conditions for a particular shear deformation theory is no longer an issue, as this MATLAB code does all that for us. In Fig. 2, it is shown the assembling procedure of the stiffness matrix on layer \(k\) for the ESL approach.

Substituting the geometrical relations, the constitutive and the Unified Formulation into the variational statement PVD, for the \(k\)th layer, one has:

\[
\sum_{k=1}^{N_L} \int_{\Omega_k} \left\{ (D_{xy} + A_{xy})\partial u_x^T (D_{xy} + A_{xy}) u_x + C_{xy} (D_{xy} + A_{xy}) u_x \right\} \delta u_x \partial u_x d\Omega_k \\
+ \left( C_{mn} (D_{xy} + A_{xy}) u_x \right) \delta u_x d\Omega_k - \frac{N_L}{k} \delta u_x d\Omega_k
\]

At this point, the formula of integration by parts is applied:

\[
\int_{\Omega_k} (D_{xy}) \partial u_x^T \partial u_x d\Omega_k = - \int_{\Omega_k} \partial u_x^T (D_{xy}) \partial u_x d\Omega_k \\
+ \int_{\Omega_k} \partial u_x^T \partial (L_{xy}) \partial u_x d\Omega_k
\]

where \(L_{xy}\) is obtained applying the Gradient theorem:

\[
\int_{\Omega_k} \frac{\partial \psi}{\partial t} d\Omega_k = \int_{\Omega_k} n_{\Omega} \psi ds
\]

being \(n_{\Omega}\) the components of the normal \(\tilde{n}\) to the boundary along the direction \(i\). After integration by parts and the substitution of \(C_{UF}\), the governing equations and boundary conditions for the shell in the mechanical case are obtained:

\[
\begin{align*}
\sum_{k=1}^{N_L} \int_{\Omega_k} \delta u_x^T \left[ -D_{xy} + A_{xy} \right] \left( C_{xy} (D_{xy} + A_{xy}) + C_{mn} (D_{xy} + A_{xy}) \right) u_x \\
+ C_{mn} (D_{xy} + A_{xy}) u_x \delta u_x d\Omega_k \\
\end{align*}
\]

where \(I_{xy}^i\) and \(I_{xy}^u\) depend on the boundary geometry:

\[
I_{xy} = \begin{bmatrix}
\frac{n_{\Omega}}{n_{\Omega}} & 0 & 0 \\
0 & \frac{n_{\Omega}}{n_{\Omega}} & 0 \\
0 & 0 & \frac{n_{\Omega}}{n_{\Omega}}
\end{bmatrix}, \\
I_{xy}^u = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The normal to the boundary of domain \(\Omega\) is:

\[
\begin{bmatrix}
\hat{n}_{x} \\
\hat{n}_{y} \\
\hat{n}_{z}
\end{bmatrix} = \begin{bmatrix}
\cos (\phi_x) \\
\cos (\phi_y)
\end{bmatrix}
\]

where \(\phi_x\) and \(\phi_y\) are the angles between the normal \(\hat{n}\) and the direction \(x\) and \(y\), respectively.

The governing equations for a multi-layered shell subjected to mechanical loadings are:

\[
\delta u_x^T \left[ K_{mn}^{\text{ext}} \right] u_x = P_{\text{ext}}
\]

where the fundamental nucleus \(K_{mn}^{\text{ext}}\) is obtained as:
\[ K_{\text{uu}}^{\text{kk}} = \int_{\partial \Omega} \left[ -[D_p + A_p]^{\text{C}}_{\text{sp}}^{\text{D}}[D_p + A_p] + -[D_p + A_p]^{\text{C}}_{\text{sp}}^{\text{a}}[D_p + A_p] \right. \\
\left. + [D_p + A_p + A_k]^{\text{C}}_{\text{sp}}^{\text{a}} + [D_p + A_p - A_k]^{\text{C}}_{\text{sp}}^{\text{a}} \right] F, F H_i^p H_j^p dz \] (18)

and the corresponding Neumann-type boundary conditions on \( \Gamma_k \) are:
\[ \Pi_k^{\text{uu}} = \Pi_k^{\text{uu}} \] (19)

where
\[ \Pi_k^{\text{uu}} = \int_{\partial \Omega} \left[ [D_p + A_p]^{\text{C}}_{\text{sp}}^{\text{D}} + [D_p + A_p + A_k]^{\text{C}}_{\text{sp}}^{\text{a}} + [D_p + A_p - A_k]^{\text{C}}_{\text{sp}}^{\text{a}} \right] F, F H_i^p H_j^p dz \] (20)

and \( p_{\text{uu}} \) are variationally consistent loads with applied pressure.

3.2. Fundamental nuclei

The following integrals are introduced to perform the explicit form of fundamental nuclei:
\[ \left( \rho^k_{\text{uu}}, \tilde{p}^k_{\text{uu}}, \tilde{p}^k_{\text{uu}}, \tilde{p}^k_{\text{uu}}, \tilde{p}^k_{\text{uu}} \right) = \int_{\partial \Omega} F, F \left( \frac{1}{H}, H, H, H, H, H \right) H_i^p H_j^p dz \] (21)

The fundamental nuclei \( K_{\text{uu}}^{\text{kk}} \) is reported for doubly curved shells (radius of curvature in both \( x \) and \( y \) directions (see Fig. 1)).

\[ K_{\text{uu}}^{\text{kk}}(11) = -C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 - C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 - C_{20}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 + C_{55}^{(j, j, j, j, j, j, j) \rho_{\text{uu}} \rho_{\text{uu}}^2} + C_{55}^{(j, j, j, j, j, j, j) \rho_{\text{uu}} \rho_{\text{uu}}^2} \] (22)

The application of boundary conditions makes use of the fundamental nuclei \( \Pi_k \) in the form:

\[ \left( \Pi_k^{\text{uu}}(1) \right)_{11} = C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 + n \left( C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 + n \left( C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 + n \left( C_{14}^{\text{kk}} \rho_{\text{uu}} \rho_{\text{uu}}^2 \right) \right) \right) \] (23)

One can note that all the equations written for the shell degenerate in those for the plate when \( \frac{1}{\rho_{\text{uu}}} = 0 \). In practice, the radii of curvature are set to 10^6.

3.3. Dynamic governing equations

The PVD for the dynamic case is expressed as:
\[ \sum_{k=1}^{N} \int_{\Omega_k} \int_{\partial \Omega_k} \left\{ \delta^k \sigma^k_{pc} \right\} d\Omega_k \, dz = \sum_{k=1}^{N} \int_{\Omega_k} \rho^k \delta^k \mathbf{u}^T \mathbf{u}^k \, d\Omega_k \, dz + \sum_{k=1}^{N} \delta^k \mathbf{a}_k \]  

(24)

where \( \rho^k \) is the mass density of the \( k \)th layer and double dots denote acceleration.

By substituting the geometrical relations, the constitutive equations and the Unified Formulation, one obtains the following governing equations:

\[
\delta^k \mathbf{u}^T : K^k_{\text{st}} \mathbf{u}^k = M^k_{\text{st}} \mathbf{u}^k + P^k_{\text{at}}
\]

(25)

In the case of free vibrations one has:

\[
\delta^k \mathbf{u}^T : K^k_{\text{st}} \mathbf{u}^k = M^k_{\text{st}} \mathbf{u}^k
\]

(26)

where \( M^k_{\text{st}} \) is the fundamental nucleus for the inertial term. The explicit form of that is:

\[
M^k_{\text{st}} = \rho^k \mathbf{I}
\]

(27)

where the meaning of the integral \( f_{\omega}^{3} \) has been illustrated in Eq. (21). The geometrical and mechanical boundary conditions are the same of the static case. Because the static case is only considered, the mass terms will be neglected.

4. The radial basis function method

4.1. The static problem

Radial basis functions (RBF) approximations are mesh-free numerical schemes that can exploit accurate representations of the boundary, are easy to implement and can be spectrally accurate. In this section the formulation of a global unsymmetrical collocation RBF-based method to compute elliptic operators is presented.

Consider a linear elliptic partial differential operator \( L \) and a bounded region \( \Omega \) in \( \mathbb{R}^n \) with some boundary \( \partial \Omega \). In the static problems, the displacements \( \mathbf{u} \) are computed from the global system of equations

\[
\mathcal{L}\mathbf{u} = \mathbf{f} \text{ in } \Omega
\]

(28)

\[
\mathcal{L}\mathbf{u} = \mathbf{g} \text{ on } \partial \Omega
\]

(29)

where \( \mathcal{L} \) and \( \mathcal{L}_b \) are linear operators in the domain and on the boundary, respectively. The right-hand side of (28) and (29) represent the external forces applied on the plate or shell and the boundary conditions, respectively along the perimeter of the plate or shell, respectively. The PDE (Partial Differential Equation) problem defined in (28) and (29) will be replaced by a finite problem, defined by an algebraic system of equations, after the radial basis expansions.

4.2. The eigenproblem

The eigenproblem looks for eigenvalues \( \lambda \) and eigenvectors \( \mathbf{u} \) that satisfy

\[
\mathcal{L}\mathbf{u} + \lambda \mathbf{u} = 0 \text{ in } \Omega
\]

(30)

\[
\mathcal{L}\mathbf{u} = 0 \text{ on } \partial \Omega
\]

(31)

As in the static problem, the eigenproblem defined in (30) and (31) is replaced by a finite-dimensional eigenvalue problem, based on RBF approximations.

4.3. Radial basis functions approximations

The radial basis function \( \psi \) approximation of a function \( \mathbf{u} \) is given by

\[
\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{N} \mathbf{a}_i \psi(||\mathbf{x} - \mathbf{x}_i||_2) \quad \mathbf{x} \in \mathbb{R}^n
\]

(32)

where \( \mathbf{x}_i, i = 1, \ldots, N \) is a finite set of distinct points (centers) in \( \mathbb{R}^n \). The most common RBFs are

Cubic : \( \psi(r) = r^2 \)

Thin plate splines : \( \psi(r) = r^2 \log(r) \)

Wendland functions : \( \psi(r) = (1 - r)^m r \)

Gaussian : \( \psi(r) = e^{-r^2} \)

Multiquadrics : \( \psi(r) = \sqrt{r^2 + c^2} \)

Inverse multiquadrics : \( \psi(r) = (c^2 + r^2)^{-1/2} \)

where the Euclidean distance \( r \) is real and non-negative and \( c \) is a positive shape parameter. Hardy [45] introduced multiquadrics in the analysis of scattered geographical data. In the 1990s Kansa [26] used multiquadrics for the solution of partial differential equations. Considering \( N \) distinct interpolations, and knowing \( \mathbf{u}(\mathbf{x}_i) \), \( j = 1, 2, \ldots, N \), one finds \( \mathbf{a}_i \) by the solution of a \( N \times N \) linear system

\[
\mathbf{A} \mathbf{a} = \mathbf{u}
\]

(33)

where \( \mathbf{A} = \left[ \psi(||\mathbf{x} - \mathbf{x}_j||_2) \right]_{i,j=1}^{N} \), \( \mathbf{z} = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N]^T \) and \( \mathbf{u} = [\mathbf{u}(\mathbf{x}_1), \mathbf{u}(\mathbf{x}_2), \ldots, \mathbf{u}(\mathbf{x}_N)]^T \).

4.4. Solution of the static problem

The solution of a static problem by radial basis functions considers \( N \) nodes in the domain and \( N_b \) nodes on the boundary, with a total number of nodes \( N = N + N_b \). One can denote the sampling points by \( \mathbf{x}_i \in \Omega, i = 1, \ldots, N \) and \( \mathbf{x}_i \in \partial \Omega, i = N_b + 1, \ldots, N \). At the points in the domain, the following system of equations is solved

\[
\sum_{i=1}^{N} \mathbf{a}_i \mathcal{L}\phi(||\mathbf{x} - \mathbf{x}_j||_2) = \mathbf{f}(\mathbf{x}_j), \quad j = 1, 2, \ldots, N_j
\]

(34)

or

\[
\mathcal{L}\mathbf{z} = \mathbf{F}
\]

(35)

where

\[
\mathbf{F} = \left[ \mathcal{L}\phi(||\mathbf{x} - \mathbf{x}_j||_2) \right]_{i,j=1}^{N_j,N}
\]

(36)

At the points on the boundary, the following boundary conditions are imposed

\[
\sum_{i=1}^{N} \mathbf{a}_i \mathcal{L}_b \phi(||\mathbf{x} - \mathbf{x}_j||_2) = \mathbf{g}(\mathbf{x}_j), \quad j = N_b + 1, \ldots, N
\]

(37)

or

\[
\mathbf{Bz} = \mathbf{G}
\]

(38)
where
\[ B = L^T \phi \left( \| x_{N_i+1} - y_j \|_2 \right)_{N_i+1} \]

Therefore, one can write a finite-dimensional static problem as
\[ \begin{bmatrix} L^T \\ B \end{bmatrix} \mathbf{x} = \begin{bmatrix} F \\ G \end{bmatrix} \tag{39} \]

By inverting the system (39), one obtains the vector \( x \). Then, the solution \( u \) is calculated using the interpolation Eq. (32).

4.5. Solution of the eigenproblem

Consider \( N \) nodes in the interior of the domain and \( N_b \) nodes on the boundary, with \( N = N + N_b \). The interpolation points are denoted by \( x_i \in \Omega, i = 1, \ldots, N \) and \( x_i = \partial \Omega, i = N + 1, \ldots, N_b \). At the points in the domain, the following eigenproblem is defined
\[ \sum_{i=1}^{N} 2_i \lambda \phi \left( \| x - y_i \|_2 \right) = \lambda \mathbf{u}(x_i), j = 1, 2, \ldots, N_i \tag{40} \]

or
\[ L^T \mathbf{x} = \lambda \mathbf{u} \tag{41} \]

where
\[ L^T = \left[ \lambda \phi \left( \| x - y_j \|_2 \right) \right]_{N_b+N} \tag{42} \]

At the points on the boundary, the imposed boundary conditions are
\[ \sum_{j=1}^{N_b} 2_j L^T \phi \left( \| x - y_j \|_2 \right) = 0, \quad j = N_i + 1, \ldots, N \tag{43} \]

or
\[ B \mathbf{x} = 0 \tag{44} \]

Eqs. (41) and (44) can now be solved as a generalized eigenvalue problem
\[ \begin{bmatrix} L^T, 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} A^T \\ 0 \end{bmatrix} \mathbf{x} \tag{45} \]

where
\[ A^T = \phi \left( \| x_{N_i} - y_j \|_2 \right)_{N_i+1} \]

4.6. Discretization of the equations of motion and boundary conditions

The radial basis collocation method follows a simple implementation procedure. Taking Eq. (39), one computes
\[ \mathbf{x} = \begin{bmatrix} L^T \\ B \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \tag{46} \]

This \( x \) vector is then used to obtain solution \( u \), by using (32). If derivatives of \( u \) are needed, such derivatives are computed as
\[ \frac{\partial \mathbf{u}}{\partial x} = \sum_{j=1}^{N} 2_j \frac{\partial \phi_j}{\partial x} \]

\[ \frac{\partial^2 \mathbf{u}}{\partial x^2} = \sum_{j=1}^{N} 2_j \frac{\partial^2 \phi_j}{\partial x^2} \]

In the present collocation approach, one needs to impose essential and natural boundary conditions. Consider, for example, the condition \( w = 0 \), on a simply supported or clamped edge. The conditions are enforced by interpolating as
\[ w = 0 \rightarrow \sum_{j=1}^{N} 2_j w_j \phi_j = 0 \tag{49} \]

Other boundary conditions are interpolated in a similar way.

4.7. Free vibrations problems

For free vibration problems, the external forces are set to zero, and an harmonic solution is assumed for the displacements \( u_0 \), \( u_1 \), \( u_2 \), \( u_3 \), etc.
\[ \begin{align*}
  u_0 &= U_0(x,y)e^{i\omega t} \\
  u_1 &= U_1(x,y)e^{i\omega t} \\
  u_2 &= U_2(x,y)e^{i\omega t} \\
  u_3 &= U_3(x,y)e^{i\omega t} 
\end{align*} \tag{50} \]

where \( \omega \) is the frequency of natural vibration. Substituting the harmonic expansion into Eq. (45) in terms of the amplitudes \( U_0, U_1, U_2, U_3 \), \( V_0, V_1, V_2, V_3 \), one can obtain the natural frequencies and vibration modes for the plate or shell problem, by solving the eigenproblem
\[ \begin{bmatrix} L^T - \omega^2 \Theta \end{bmatrix} X = 0 \tag{53} \]

where \( L^T \) collects all stiffness terms and \( \Theta \) collects all terms related to the inertial terms. In (53) \( X \) are the modes of vibration associated with the natural frequencies defined as \( \omega \).

5. Numerical examples

All numerical examples consider a Chebyshev grid (see Fig. 3) and a Wendland function, defined as
\[ \phi(r) = (1 - cr)^6 \left( 32(cr)^3 + 25(cr)^2 + 8cr + 1 \right) \tag{54} \]

where the shape parameter \( c \) was obtained by an optimization procedure, as detailed in Ferreira and Fasshauer [46].

5.1. Spherical shell in bending

A laminated composite spherical shell is here considered, of side \( a \) and thickness \( h \), composed of layers oriented at \( [0^\circ/90^\circ/0^\circ] \).

![Fig. 3. A sketch of a Chebyshev grid for 13 x 13 points.](image-url)
The in-plane displacements, the transverse displacements, the normal stresses and the in-plane and transverse shear stresses are presented in normalized form as

\[ w = \frac{10^3 W_{0.2,2,2;0,h} h^3 E_2}{P a^4}, \quad \sigma_{xx} = \frac{\sigma_{y0.2,2,2;0,h} h^2}{P a^2}, \quad \sigma_{yy} = \frac{\sigma_{y0.2,2,2;0,h} h^2}{P a^2}, \quad \tau_{xz} = \frac{\tau_{xz0.2,2,0,h}}{P a}, \quad \tau_{xy} = \frac{\tau_{x0.2,0,0,h} h^2}{P a^2} \]

The shell is simply supported on all edges.

In Table 1, the static deflections for the present shell model are compared with results of Reddy shell formulation using first-order and third-order shear deformation theories [47]. Nodal grids with 13 × 13, 17 × 17, and 21 × 21 points are considered. Various values of \( h/a \) and two values of \( h/a \) (10 and 100) are taken for the analysis. Results are in good agreement for various \( h/a \) ratios with the higher-order results of Reddy [47].
who considered both the first-order (FSDT) and the third-order (HSDT) theories. Present results are compared with analytical solutions by Reddy and Liu [47], who considered both the First-order Shear Deformation Theory (FSDT) and the High-order Shear Deformation Theory (HSDT). The first-order theory overpredicts the fundamental natural frequencies of symmetric thick shells and symmetric shallow thin shells. The present radial basis function method is compared with analytical results by Reddy [47] and shows excellent agreement.

Table 4 contain nondimensionalized natural frequencies obtained using the present SSDT theory for cross-ply cylindrical shells with lamination schemes [0/90/0], [0/90/90/0]. Present results are compared with analytical solutions by Reddy and Liu [47] who considered both the first-order (FSDT) and the third-order (HSDT) theories. The present radial basis function method is compared with analytical results by Reddy [47] and shows excellent agreement.

Table 4

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<th>R/a</th>
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<th>[0/90/90/0] a/h = 100</th>
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5.2. Free vibration of spherical and cylindrical laminated shells

Nodal grids with $13 \times 13$, $17 \times 17$, and $21 \times 21$ points are considered. In Tables 2 and 3 the nondimensionalized natural frequencies from the present SSDT theory for various cross-ply spherical shells are compared with analytical solutions by Reddy and Liu [47], who considered both the First-order Shear Deformation Theory (FSDT) and the High-order Shear Deformation Theory (HSDT). The first-order theory overpredicts the fundamental natural frequencies of symmetric thick shells and symmetric shallow thin shells. The present radial basis function method is compared with analytical results by Reddy [47] and shows excellent agreement.

Table 4 contains nondimensionalized fundamental frequencies of cross-ply spherical shells, $\omega = \omega C_{176} \sqrt{p/E_2}$, using a grid of $13 \times 13$ points, for $a/h = 100, R/a = 10$.

![Fig. 4](image-url) Fig. 4. First four vibrational modes of cross-ply laminated spherical shells, $\omega = \omega C_{176} \sqrt{p/E_2}$, laminate ([0/90/90/0]), grid $13 \times 13$ points, $a/h = 100, R/a = 10$.

6. Concluding remarks

In this paper a sinusoidal shear deformation theory was implemented for the finite element for laminated orthotropic elastic shells through a multi-quadratics discretization of equations of motion and boundary conditions. The multi-quadratics radial basis function method for the solution of shell bending and free vibration problems was presented. Results for static deformations and natural frequencies were obtained and compared with other sources. This meshless approach demonstrated that is very successful in the static deformations and free vibration analysis of laminated composites.
ite shells. Advantages of radial basis functions are absence of mesh, ease of discretization of boundary conditions and equations of equilibrium or motion and very easy coding. The static displacements and the natural frequencies obtained from present method are shown to be in excellent agreement with analytical solutions.

Acknowledgements

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References