

Evaluation of Hierarchical Vector Basis Functions for Quadrilateral Cells

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New hierarchical vector basis functions for quadrilateral cells are introduced, and the matrix condition numbers associated with their use are compared to those of existing vector basis families to assess the relative linear independence of the functions. Scale factors are employed to improve the condition numbers. In addition, the proper use of subsets of these families to transition from one order to another (as needed for adaptive p -refinement) without exciting spurious modes is considered.

Index Terms—Boundary elements, Helmholtz equation, hierarchical basis functions, vector finite elements.

I. INTRODUCTION

VARIOUS hierarchical vector basis functions have been proposed for use with the finite-element (FE) analysis of electromagnetic field problems. The present contribution focuses on curl-conforming bases for quadrilateral-cell discretizations of the vector Helmholtz equation

$$\nabla \times \nabla \times \bar{H} = k^2 \bar{H}. \quad (1)$$

References [1]–[5] present several families of existing functions of this type. The p th order basis set from each family spans the identical mixed-order space of Nedelec [6], but hierarchical functions usually suffer from a loss of linear independence as their order increases. Thus, one family might be superior to another in that regard.

In this study, we assess the relative linear independence of this type of basis by comparing the matrix condition numbers arising from the element and global systems associated with entries of the form

$$T_{mn} = \int \int \bar{B}_m \bullet \bar{B}_n dx dy \quad (2)$$

where \bar{B}_n denotes a vector basis function. Our recent research suggests that this condition number usually correlates with the conditioning of the overall system arising from the FE discretization of the Helmholtz equation and with the performance of iterative algorithms often used to solve those systems. However, such a comparison is misleading because most of the published families of basis functions do not incorporate scale factors, and the use of such factors can greatly improve the conditioning. Thus, we first identify suitable scale factors in an attempt to make a fair comparison between the various families. A similar study was recently completed for triangular-cell vector bases [7], and demonstrated that scale factors can improve the matrix conditioning by many orders of magnitude. The introduction of scale factors has an effect similar to diagonal matrix preconditioning prior to iterative solution.

We include in our comparison a new set of hierarchical vector bases designed for linear independence [8], [9].

Generally, hierarchical basis functions are intended for use within an adaptive p -refinement algorithm. Such an implementation requires the transition between low-order representations over some part of the domain and higher order representations over some other region. However, poorly constructed transition schemes can excite spurious eigenvalues and eigenfunctions. The proper construction of transition functions is discussed.

II. NEW BASES FOR QUADRILATERAL CELLS

All the basis families in [1]–[5] and [8], [9] employ the same polynomial degrees of freedom, but each family uses a different specific realization. The new functions are constructed from Legendre (P_n) and Jacobi ($P_n^{(2,2)}$) polynomials, defined

$$\begin{cases} P_0(z) = 1 \\ P_1(z) = z \end{cases} \quad (3)$$

$$\begin{cases} P_0^{(2,2)}(z) = 1 \\ P_1^{(2,2)}(z) = 3z \end{cases} \quad (4)$$

and satisfying the recurrences

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z) \quad (5)$$

$$(n+1)(n+5)P_{n+1}^{(2,2)}(z) = (n+3) \left[(2n+5)zP_n^{(2,2)}(z) - (n+2)P_{n-1}^{(2,2)}(z) \right]. \quad (6)$$

Using these polynomials, we define

$$E_n(z) = \sqrt{2n+1}P_n(z) \quad (7)$$

$$f_n(z) = \sqrt{\frac{(2n+5)(n+3)(n+4)}{3(n+1)(n+2)}}P_n^{(2,2)}(z). \quad (8)$$

On a reference cell in (u, v) coordinates, where $-1 < u, v < 1$, edge based functions of order p have the form

$$\bar{B}_k^e = \hat{u}(1-v)E_k(u)/4 \text{ for } k = 0, 1, \dots, p \quad (9)$$

while the face-based (or cell-based) functions have the form

$$\begin{aligned} \bar{B}_{ik}^f &= \hat{u}(1-v^2)f_{i-1}(v)E_k(u)/8 \\ &\text{for } i = p; k = 0, 1, \dots, p-1 \\ &\text{and } k = p; i = 1, 2, \dots, p. \end{aligned} \quad (10)$$

The edge-based functions straddle adjacent cells and maintain tangential-vector continuity between cells; the face-based functions are entirely local to a given cell. Each successive order

uses four additional edge functions and $4p$ additional face functions. The motivation behind the development of these bases is described in a forthcoming publication [9].

The set of 24 quadratic-tangential/cubic-normal (QT/CuN) bases is given as follows. The four constant-tangential, linear-normal (CT/LN) base vectors (order 0) are

$$\bar{B}_1 = \hat{v}(1 - u) \quad (11)$$

$$\bar{B}_2 = \hat{v}(1 + u) \quad (12)$$

$$\bar{B}_3 = \hat{u}(1 - v) \quad (13)$$

$$\bar{B}_4 = \hat{u}(1 + v). \quad (14)$$

To expand the space to the next order, we first consider four bilinear edge-based functions given by

$$\bar{B}_5 = \hat{v}\sqrt{3}(1 - u)v \quad (15)$$

$$\bar{B}_6 = \hat{v}\sqrt{3}(1 + u)v \quad (16)$$

$$\bar{B}_7 = \hat{u}\sqrt{3}u(1 - v) \quad (17)$$

$$\bar{B}_8 = \hat{u}\sqrt{3}u(1 + v). \quad (18)$$

To reach Nedelec's mixed order 1, four additional face-based functions are required. These have the form

$$\bar{B}_9 = \hat{v}\frac{\sqrt{10}}{2}(1 - u^2) \quad (19)$$

$$\bar{B}_{10} = \hat{v}\frac{\sqrt{30}}{2}(1 - u^2)v \quad (20)$$

$$\bar{B}_{11} = \hat{u}\frac{\sqrt{10}}{2}(1 - v^2) \quad (21)$$

$$\bar{B}_{12} = \hat{u}\frac{\sqrt{30}}{2}(1 - v^2)u. \quad (22)$$

The preceding set of 12 functions spans the linear-tangential, quadratic normal (LT/QN) space, here denoted order 1. To expand the space to order 2, or QT/CuN, we add the four edge-based functions

$$\bar{B}_{13} = \hat{v}\sqrt{5}(1 - u)\left(\frac{3v^2 - 1}{2}\right) \quad (23)$$

$$\bar{B}_{14} = \hat{v}\sqrt{5}(1 + u)\left(\frac{3v^2 - 1}{2}\right) \quad (24)$$

$$\bar{B}_{15} = \hat{u}\sqrt{5}\left(\frac{3u^2 - 1}{2}\right)(1 - v) \quad (25)$$

$$\bar{B}_{16} = \hat{u}\sqrt{5}\left(\frac{3u^2 - 1}{2}\right)(1 + v) \quad (26)$$

and augment this set with eight face-based functions

$$\bar{B}_{17} = \hat{v}\frac{\sqrt{50}}{4}(1 - u^2)(3v^2 - 1) \quad (27)$$

$$\bar{B}_{18} = \hat{v}\frac{\sqrt{70}}{2}(1 - u^2)u \quad (28)$$

$$\bar{B}_{19} = \hat{u}\frac{\sqrt{50}}{4}(1 - v^2)(3u^2 - 1) \quad (29)$$

$$\bar{B}_{20} = \hat{u}\frac{\sqrt{70}}{2}(1 - v^2)v \quad (30)$$

$$\bar{B}_{21} = \hat{v}\frac{\sqrt{210}}{2}(1 - u^2)uv \quad (31)$$

$$\bar{B}_{22} = \hat{v}\frac{\sqrt{350}}{4}(1 - u^2)u(3v^2 - 1) \quad (32)$$

$$\bar{B}_{23} = \hat{u}\frac{\sqrt{210}}{2}(1 - v^2)uv \quad (33)$$

$$\bar{B}_{24} = \hat{u}\frac{\sqrt{350}}{4}(1 - v^2)v(3u^2 - 1). \quad (34)$$

This set of 24 functions spans the QT/CuN space for quadrilateral cells. These functions will maintain curl-conforming properties on arbitrary quadrilateral or curved cells defined by mappings $x(u, v)$, $y(u, v)$ if they are mapped using the transformation

$$\begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} B_u \\ B_v \end{bmatrix}. \quad (35)$$

III. CONDITION NUMBER COMPARISON

To evaluate the relative linear independence associated with the various basis families, the condition number of the global Gram matrix in (2) was evaluated for a variety of quadrilateral-cell models. Since individual members of each hierarchical family can be scaled in an almost arbitrary way, and most authors present their bases without scale factors, to ensure a fairer comparison we introduce scale factors that improve the matrix condition numbers for each family.

As in [7], a relatively coarse scaling was developed, as follows. The four base vectors (order 0) were not scaled, other than to ensure tangential-vector continuity between cells. The next four edge-based functions were scaled with a single, common factor. The factor was real-valued and chosen to minimize the 8 by 8 element matrix condition number for a square cell. The remaining four face-based functions needed to obtain an order-one representation were also scaled with a single common factor, obtained to minimize the matrix condition number for the 12 by 12 element matrix. This process was continued as the element order increased, yielding a single scale factor for the four additional edge bases of each order and a single scale factor for the group of face bases of each order. The bases of Jorgensen *et al.* [4] and the new bases presented above contain built-in scale factors; the additional scale factors are multiplied with those factors. For the basis families under consideration, our coarse scaling was always able to improve the element matrix conditioning over the original form of the published basis functions. However, when applied to actual quadrilateral-cell meshes, in some cases the original bases outperformed the scaled set.

The scale factors obtained by this process are clearly not the best possible factors for a general mesh. Furthermore, one could assign an independent factor to each basis function and employ a more sophisticated optimization; for simplicity, we do not investigate such an approach here. We also point out that in the comparison shown below, the Ainsworth bases [1] and the Jorgensen bases [4] involve the identical functions up to QT/CuN order, except for the scale factors introduced by Jorgensen, yet the resulting condition numbers are substantially different.

As a representative example, Table I shows the matrix condition numbers arising from an 18-cell model of a 2:1 rectangular

TABLE I
CONDITION NUMBER COMPARISON

Condition numbers of the order-351 T-matrix arising from an 18-cell model of a 2:1 rectangular cavity, constructed with skewed quadrilateral cells, for QT/CuN bases		
basis family	condition number	
	unscaled	scaled
Graglia, Wilton, Peterson [10]	1256	
Ainsworth & Coyle [1]	7084	1627
Ilic & Notaros [2]	18614	3191
Djordjevic & Notaros (Chebyshev/Chebyshev) [3]	1512	1937
Jorgensen et al. [4]	490	482
Zaglmayr [5]	2196	796
Graglia & Peterson (section II)	508	482

cavity, constructed from a 6 by 3 model with interior nodes irregularly located to produce skewed quadrilateral cells. This example used the set of 24 QT/CuN bases from each family. Condition numbers for the original or “unscaled” bases and those obtained with the coarse scaling described above are presented. Results from the interpolatory bases of [10] are also shown for comparison. All basis families were also tested to ensure that they produce identical results for cavity eigenvalues, and therefore are correctly implemented.

Table I suggests that the basis functions of Jorgensen *et al.* [4] and the new basis set proposed above outperform the other hierarchical families in terms of condition number, and this was always the case in the tests we conducted on a variety of meshes. It is perhaps significant that these are the two basis families that were originally proposed with specific scale factors.

IV. TRANSITIONING STRATEGY

Hierarchical bases are intended for use in adaptive p -refinement schemes, where the basis order is varied across the mesh to compensate for varying error levels due to the presence of sources, geometrical features, material density changes, etc. However, since one of the major issues with the vector FE method is the appearance of spurious modes, the transition from one polynomial order to another must be done with care to avoid this problem.

Although it is implied in the literature that any combination of hierarchical basis functions can be used within a particular cell, such an approach is unlikely to avoid spurious modes. Instead, we consider two fairly restrictive approaches for transitioning in 2D. These make use of the fact that the basis functions are either edge-based functions that straddle two adjacent cells, or are face-based functions that are entirely confined to a single quadrilateral cell. In both approaches, it is assumed that the order of the edge-based functions can vary from edge to edge (perhaps with some constraints, such as a limit of two consecutive orders per cell). In the first approach (scheme 1), the cell-based functions in each cell are assigned to be complete to the lowest order of the surrounding edges. (Here, we use the term complete in the context of Nedelec’s mixed-order reduced-gradient spaces [6].) In other words, if a cell has two

TABLE II
EIGENVALUE COMPARISON

Lowest nonzero cutoff wavenumbers for a 2:1 rectangular cavity, for two transition schemes and two different representation orders.				
Scheme 1	Scheme 1	Scheme 2	Scheme 2	Exact
“0-1-0”	“1-2-1”	“0-1-0”	“1-2-1”	
6.62	7.02	6.99	7.02	7.02
7.38 *	8.57 *			
7.46 *	8.62 *			
7.75 *	8.74 *			
8.43 *	8.76 *			
8.47 *	8.82 *			
8.78 *	8.84 *			
9.67	8.93	8.94	8.89	8.89
11.89	11.35	11.58	11.34	11.33
14.13	13.06	13.00	12.96	12.95
15.84	14.11	14.21	14.06	14.05
16.11	14.20	14.33	14.11	14.05
17.26	15.79	16.15	15.75	15.71
...

edges that are assigned to order p and two edges that are assigned to order $p + 1$, the full set of cell-based functions of order p are used in that cell. The alternate approach (scheme 2) is to employ cell-based functions of the highest order of the surrounding edges in each cell (in the preceding illustration, the full set of order $p + 1$ cell-based bases would be used).

As a numerical illustration of this issue, consider a 2:1 rectangular cavity represented by 18 uniform rectangular cells (three rows and six columns). All the edges surrounding the six cells in columns 3 and 4 of the rectangular domain are set to a higher order than the other edges. We consider transitioning from an order-zero representation on the sides to an order-one representation in the center cells (“0-1-0”), as well as a transition from an order-one representation on the sides to an order-two representation in the center cells (“1-2-1”). To implement scheme 1, the cell-based bases in the six middle cells are set to the higher order. To implement scheme 2, the cell-based functions in the cells immediately adjacent to those are also raised to that higher order.

Table II shows the lowest nonzero eigenvalues obtained from a solution of the curl-curl form of the vector Helmholtz equation for the magnetic field, with PEC boundaries (homogeneous Neumann boundary conditions), for the two schemes outlined above, and the Graglia and Peterson basis functions of Section II (all the other families produce the same results, except for the Zaglmayr family). These eigenvalues are the cutoff wavenumbers for the cavity, and the exact values are shown for comparison. It is immediately observed that scheme 1 produces six spurious eigenvalues for this mesh (flagged with an asterisk), while scheme 2 does not. (It was verified that the nullspace of the system was reduced in dimension by exactly six.) It appears that there is one spurious eigenvalue for each edge basis that protrudes into a cell of lower internal order. Apparently, the

face-based functions of order p provide the proper representation within a cell even in the absence of one or more edge-based functions, but an edge-based function of order p protruding into a cell of lower order excites a spurious mode.

We further observe that, with the exception of the Zaglmayr bases, none of the basis families considered above use explicit gradient functions as edge-based functions. (This is in contrast to the typical bases proposed for triangular or tetrahedral cells, where most of the proposed hierarchical edge-based bases are gradient functions.) The use of explicit gradients eliminates the spurious mode problem arising from protruding edge bases; for instance, our tests confirmed that the Zaglmayr bases can be used with either scheme 1 or scheme 2 without producing spurious modes. As an alternative, most of the other families' edge functions can be easily converted into gradients if desired.

V. CONCLUSION

A new set of hierarchical vector basis functions for quadrilateral cells is proposed. These basis functions compare favorably to other families in terms of their linear independence, as evaluated by a comparison of matrix condition numbers. The impact of scale factors on matrix conditioning was illustrated by example. Finally, procedures for transitioning from one order to another, as within a p -refinement implementation, were evaluated. For most of the quadrilateral-cell basis families, allowing an edge basis to protrude into a cell of lower order will lead to spurious modes. To avoid spurious modes, the face-based functions within each cell should be complete to the highest order of that cell's edge-based functions.

The condition number is one characteristic of a basis; further testing of the new basis functions is necessary to explore other aspects of their performance (such as the convergence rate of iterative solvers) for realistic problems.

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