Comparison of hazard rates for dependent random variables

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Abstract

In this paper we define and study a new notion for the comparison of the hazard rates of two random variables taking into account their mutual dependence. Properties, applications and the comparison for a data set are given.

Keywords: Joint stochastic orders, hazard rate, copula.

1 Introduction and motivation

Let X and Y be the random lifetimes of two individuals subjected to a common random environment and conditionally independent on it. Let Θ be the random environment, with $\Theta \sim Bernoulli(1/2)$. Assume that

$$[X|\Theta = 0] \sim Exp(0.6) \qquad [Y|\Theta = 0] \sim Exp(1.2)$$

and

$$[X|\Theta = 1] \sim Exp(2) \qquad [Y|\Theta = 1] \sim Exp(0.5),$$

where $[X|A]$ and $[Y|A]$ denote the lifetimes having as distributions those of X and Y, respectively, conditioned on the event A.

Let us denote by F_X, F_Y and F the marginal distribution functions of X and Y and their joint distribution function. The joint survival $\overline{F}$ of (X, Y) is

$$\overline{F}(x, y) = P[X > x, Y > y] = \frac{1}{2}\exp\big(-0.6x - 1.2y\big) + \frac{1}{2}\exp\big(-2x - 0.5y\big),$$

while the univariate marginal survival functions $\overline{F}_X$ and $\overline{F}_Y$ are

$$\overline{F}_X(x) = P[X > x] = \frac{1}{2}\big(\exp(-0.6x) + \exp(-2x)\big)$$

and

$$\overline{F}_Y(y) = P[Y > y] = \frac{1}{2}\big(\exp(-1.2y) + \exp(-0.5y)\big).$$

To study if the residual lifetime of Y is stochastically bigger than the residual lifetime of X along the time, we have to verify if

$$X_t = [X - t|X > t] \leq_{\text{st}} [Y - t|Y > t] = Y_t, \quad \forall t > 0, \tag{1.1}$$

where $\leq_{\text{st}}$ denotes the usual stochastic order (whose definition is recalled at the end of this section).

It is well known (see Shaked and Shanthikumar, 2007) that (1.1) is equivalent to the comparison of the hazard rates, $r(t) = \frac{\partial F_X(t)}{\partial t}/\overline{F}_X(t)$ and $s(t) = \frac{\partial F_Y(t)}{\partial t}/\overline{F}_Y(t)$ of X and Y, respectively. In particular (1.1) is equivalent to $r(t) \geq s(t)$ for all $t \geq 0$. Thus we have to compare the hazard rates of X and Y, based on their marginal distributions. It can be verified that for X and Y defined as above the two hazard rates do not cross (see Figure 1), therefore $r(t) \geq s(t)$ for all $t > 0$, and we can assert that X is smaller, in the *hazard rate order*, than Y (shortly, $X \leq_{\text{hr}} Y$ or $\overline{F}_X \leq_{\text{hr}} \overline{F}_Y$).

However, if the two individuals are subject to the same environment and they age together, we should consider their lifetimes jointly, in order to take into account the dependence among the two lifetimes, that is, we should consider the comparisons

$$\widetilde{X}_t = [X - t|X > t, Y > t] \leq_{\text{st}} [Y - t|X > t, Y > t] = \widetilde{Y}_t, \quad \forall t > 0. \tag{1.2}$$

This inequality is no more verified (see Figure 1). In fact, for example, it can be observed that, for $t = 0.7$ and $s = 3$, it holds

$$P[\widetilde{X}_t > s] = \frac{\overline{F}(t+s,t)}{\overline{F}(t,t)} = 0.1034 > 0.1017 = \frac{\overline{F}(t,t+s)}{\overline{F}(t,t)} = P[\widetilde{Y}_t > s],$$

while, for example, $P[\widetilde{X}_t > s] = 0.1937 < 0.1960 = P[\widetilde{Y}_t > s]$ for $t = 0.7$ and $s = 2$.

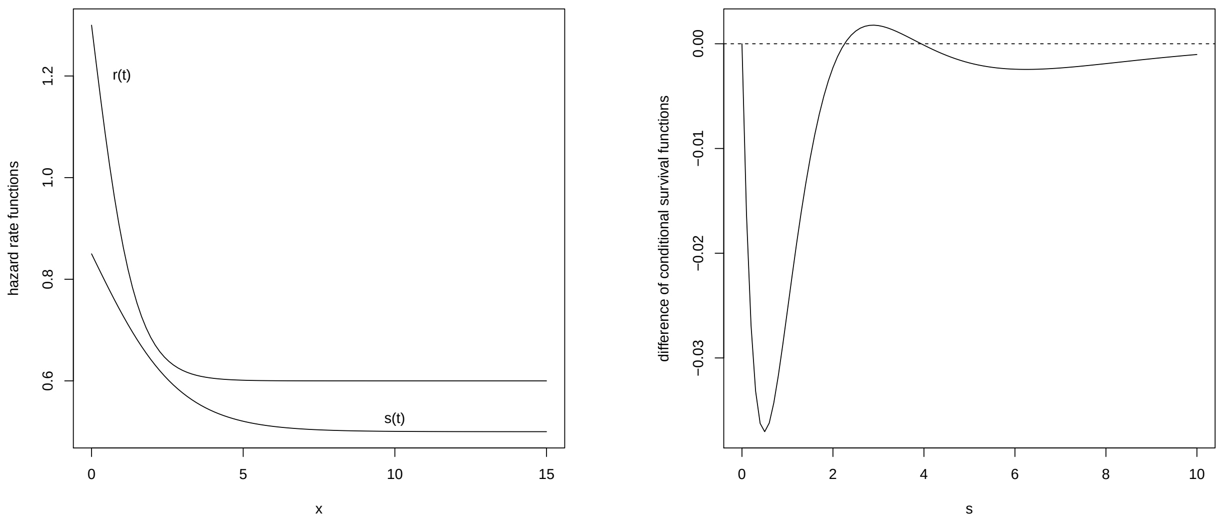

Figure 1: *Plot of $r(t)$ and $s(t)$ on the left side and the difference of the survival functions of $\widetilde{X}_t$ and $\widetilde{Y}_t$ on the right side, for $t = 0.7$.*

Furthermore, it can happen the opposite. Consider the following case. Again, let X and Y be the random lifetimes of two individuals subjected to a common random environment and conditionally independent on it. Let Θ be the random environment, with $\Theta \sim Bernoulli(1/2)$, but now assume that

$$[X|\Theta = 0] \sim Exp(30) \qquad [Y|\Theta = 0] \sim Exp(10),$$

and

$$[X|\Theta = 1] \sim Exp(2) \qquad [Y|\Theta = 1] \sim Exp(1).$$

The joint survival $\overline{F}$ of (X, Y) is

$$\overline{F}(x,y) = \frac{1}{2}\exp\big(-30x - 10y\big) + \frac{1}{2}\exp\big(-2x - y\big),$$

while the univariate marginal survival functions $\overline{F}_X$ and $\overline{F}_Y$ are

$$\overline{F}_X(x) = \frac{1}{2}\big(\exp(-30x) + \exp(-2x)\big) \quad \text{and} \quad \overline{F}_Y(y) = \frac{1}{2}\big(\exp(-10y) + \exp(-y)\big).$$

In this case we have that the two hazard rates have two crossing points (see Figure 2), thus X and Y are not ordered in the hazard rate order. However, if they age together, we have that (1.2) is satisfied, as it can been verified through Proposition 3.2, stated in Section 3, thus the lifetime Y is greater than the lifetime X in this sense.

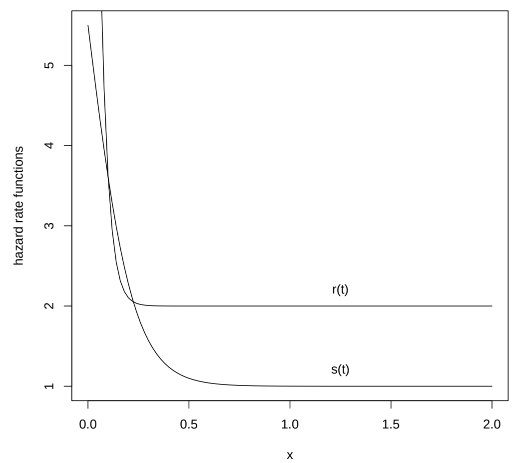

Figure 2: *Plots of* $r(t)$ *and* $s(t)$.

To sum up, if we consider individuals, or items, with non independent lifetimes, and we want to uniformly compare their residual lifetimes, then we should consider the comparison (1.2) instead of (1.1). The idea of comparisons of random variables taking into account their mutual dependence is not new; in fact, it was initiated by Shanthikumar and Yao (1991) and Shanthikumar, Yamazaki and Sakasegawa (1991), and continued later by Aly and Kochar (1993) and Righter and Shanthikumar (1993), among others, where these comparisons are usually called joint stochastic orders. Recently, these criteria have been used in several optimization problems like in analysis of random utility models, in allocation of redundant components and in portfolio selection; see Belzunce et al. (2007), (2011) and (2013), Li and You (2014) and Cai and Wei (2014).

The purpose of this paper is to introduce and study a new notion of joint hazard rate order based on the condition (1.2) for a general bivariate random vector (X, Y). The organization of this paper is the following. In Section 2 we formally define the new notion of joint hazard rate order and we show the relationships among the new notion and the ones defined and studied in previous literature. In Section 3 we provide some characterizations and properties of this joint order, and in Section 4 we describe an application to a real data set. To finish, some additional comments and remarks are pointed out in Section 5.

Along this paper the following notation is adopted. The terms "increasing" and "decreasing" are used in place of "nondecreasing" and "nonincreasing", respectively. For any random variable or random vector X and an event A, the notation $[X|A]$ describes the random variable whose distribution is the conditional distribution of X given A. Given a univariate or bivariate distribution function F, the corresponding univariate or bivariate survival function is denoted $\overline{F}$. Given a subset $A \subseteq \mathbb{R}^2$, the notation $\mathbf{1}\{(u,v) \in A\}$ is used for the indicator function on the set A.

To conclude this section, we recall some notions that will be used in the next sections. Let $\boldsymbol{x} = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$ and $\boldsymbol{x}' = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ be two vectors in $\mathbb{R}^n$. We say that $\mathbf{x}'$ is a *simple transposition* of $\mathbf{x}$, denoted by $\mathbf{x}' \leq^t \mathbf{x}$, if $x_i < x_j$, for all $i < j \in \{1, \ldots, n\}$. Given a function $f : \mathbb{R}^n \mapsto \mathbb{R}$, it is said to be *arrangement increasing (decreasing)*, denoted by $f \in AI(AD)$, if $f(\mathbf{x}) \leq (\geq) f(\mathbf{x}')$, where $\mathbf{x}'$ is a simple transposition of $\mathbf{x}$. To finish, we provide the definition of the well known usual stochastic order. For it, recall that a set $U \in \mathbb{R}^n$ is called *upper set* if $\mathbf{y} \in U$ whenever $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{x} \in U$, where $\mathbf{y} \geq \mathbf{x}$ is equivalent to $y_i \geq x_i$ for all $i = 1, .., n$. Now, given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, we say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *usual stochastic order*, denoted by $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$, if

$$P[\mathbf{X} \in U] \leq P[\mathbf{Y} \in U] \text{ for all upper sets } U \subseteq \mathbb{R}^n.$$

Observe that, in the univariate case, given two random variables X and Y it holds $X \leq_{\text{st}} Y$ if, and only if, $\overline{F}(t) \leq \overline{G}(t)$, for all $t \in \mathbb{R}$, where $\overline{F}$ and $\overline{G}$ are the survival functions of X and Y, respectively (see Shaked and Shanthinkumar, 2007, for details).

2 A new definition of hazard rate order and its relationships with previous notions

Following the ideas described in the introduction, next we define a new notion useful to compare dependent random variables.

Definition 2.1. *Let (X, Y) be a bivariate random vector, with joint survival function $\overline{F}$. We say that X is smaller than Y in the joint weak hazard rate order, denoted by $X \leq_{\text{hr:wj}} Y$, if*

$$\widetilde{X}_t = [X - t | X > t, Y > t] \leq_{\text{st}} [Y - t | X > t, Y > t] = \widetilde{Y}_t, \quad \forall t \in \mathbb{R}, \text{ such that } \overline{F}(t,t) > 0,$$

or equivalently, if

$$\overline{F}(x,y) \geq \overline{F}(y,x) \quad \forall x \leq y, \text{ such that } \overline{F}(x,x) > 0, \tag{2.1}$$

that is, if the survival function $\overline{F}$ is arrangement decreasing.

From (2.1), we observe that the $\leq_{\text{hr:wj}}$ order has been implicitly considered and applied in Aly and Kochar (1993) and, more recently, in Belzunce et al. (2011), (2013) and Cai and Wei (2014). In the first references the inequality (2.1) does not have a name, while in last reference this inequality corresponds to the UOAI notion for the vector (X,Y) and it is considered as a dependence notion. In this paper we want to focus on this notion as a criteria to compare random variables taking into account their dependence. We also want to point out that the notation of joint weak hazard rate follows the previous notation by Shanthikumar and Yao (1991). Next we recall the definition of some of these previous notions and we study the relationships among the new one and the previous ones.

First we defined three useful sets of functions.

Definition 2.2. *Let us denote by D the set of functions $D = \{g \mid g : \mathbb{R}^2 \mapsto \mathbb{R}\}$. We consider the following sets:*

a) $G_{\rm lr} = \{g \in D : g(u,v) \geq g(v,u), \text{ for all } u \leq v\}$.

b) $G_{\rm hr} = \{g \in D : g(u,v) - g(v,u), \text{ is increasing in } v, \text{ for all } u \leq v\}$.

c) $G_{\rm st} = \{g \in D : g(u,v) - g(v,u), \text{ is increasing in } v, \text{ for all } u\}$.

We recall now the definitions of the joint orders introduced and studied in Shanthikumar and Yao (1991).

Definition 2.3. *Let us consider a bivariate random vector (X,Y) with joint distribution function F, we say that*

a) X is smaller than Y in the joint likelihood ratio order, denoted by $X \leq_{\rm lr:j} Y$, if

$$E[g(X,Y)] \geq E[g(Y,X)], \text{ for all } g \in G_{\rm lr},$$

or equivalently, if (X,Y) has a joint density or a joint probability mass function $f \in G_{\rm lr}$. That is, if f is arrangement decreasing.

b) X is smaller than Y in the joint hazard rate order, denoted by $X \leq_{\rm hr:j} Y$, if

$$E[g(X,Y)] \geq E[g(Y,X)], \text{ for all } g \in G_{\rm hr},$$

or equivalently if

$$\overline{F}(x,y) - \overline{F}(y,x) \text{ is decreasing in } x \text{ , where } x \leq y. \tag{2.2}$$

c) X is smaller than Y in the joint stochastic, denoted by $X \leq_{\rm st:j} Y$, if

$$E[g(X,Y)] \geq E[g(Y,X)], \text{ for all } g \in G_{\rm st},$$

or equivalently if $(X,-Y) \leq_{\rm st} (Y,-X)$.

Another notion of joint hazard rate order, not recalled here, was provided by Aly and Kochar (1993).

Next we study the relationships among the hr:j, hr:wj and st:j orders. First, we show that the joint hazard rate order is stronger than the weak one.

Proposition 2.1. *Let us consider a bivariate random vector* (X, Y), *if* $X \leq_{\text{hr:j}} Y$ *then* $X \leq_{\text{hr:wj}} Y$.

Proof. The proof in the absolutely continuous case is given in Proposition 2.11 in Belzunce et at. (2011). For general random variables the proof can be given observing that the function $g(u, v) = \mathbf{1}_{(u>x,v>y)}$, for $y \geq x$, belongs to the set of functions G_{hr}, therefore, if $X \leq_{\text{hr:j}} Y$ then $X \leq_{\text{hr:wj}} Y$. □

Note that the opposite is not true, as shown in Cai and Wei (2014) with a counterexample at the end of Section 4. Despite, it is possible to give a characterization of the strong one in terms of the weak one, as stated in subsequent Theorem 3.2.

The weak joint hazard order satisfies properties similar to those of the strong one. For example, it is interesting to observe, and easy to verify, that the strong joint hazard order satisfies

$$X \leq_{\text{hr:j}} Y \iff \widetilde{X}_t \leq_{\text{hr:j}} \widetilde{Y}_t, \quad \forall t \text{ such that } \overline{F}(t,t) > 0. \tag{2.3}$$

The following statement shows a similar property for the weak joint hazard order.

Proposition 2.2. *Let us consider a bivariate random vector* (X, Y) *with joint survival function* $\overline{F}$. *Then we have the following equivalence*

$$X \leq_{\text{hr:wj}} Y \iff \widetilde{X}_t \leq_{\text{hr:wj}} \widetilde{Y}_t, \quad \forall t \textit{ such that } \overline{F}(t,t) > 0. \tag{2.4}$$

Proof. Since the reverse implication is obvious, let us prove the direct one. Fix $t \in \mathbb{R}$ such that $\overline{F}(t,t) > 0$, and denote by $\overline{F}_t(x,y) = \frac{\overline{F}(x+t,y+t)}{\overline{F}(t,t)}$ the joint survival function of the random vector $(\widetilde{X}_t, \widetilde{Y}_t) = [X - t, Y - t| \ X, Y > t]$. Clearly, if $X \leq_{\text{hr:wj}} Y$, or equivalently, if $\overline{F}(x,y) - \overline{F}(y,x) \geq 0$ for all $x \leq y$, then $\overline{F}(x+t, y+t) - \overline{F}(y+t, x+t) \geq 0$ for all $x \leq y$. Thus

$$\overline{F}_t(x,y) = \frac{\overline{F}(x+t,y+t)}{\overline{F}(t,t)} \geq \frac{\overline{F}(y+t,x+t)}{\overline{F}(t,t)} = \overline{F}_t(y,x), \quad \forall x \leq y.$$

□

Moreover we observe that, by definition, $X \leq_{\text{hr:wj}} Y \iff \widetilde{X}_t \leq_{\text{st}} \widetilde{Y}_t$, $\forall\, t$ such that $\overline{F}(t,t) > 0$. It is interesting to observe that we have something similar for the stronger $\leq_{\text{hr:j}}$ order, which is described in the following statement. It provides a characterization of the joint hazard rate order in terms of the joint stochastic order.

Theorem 2.1. *Let us consider a bivariate random vector (X, Y) with joint survival function $\overline{F}$. Then we have the following equivalence*

$$X \leq_{\text{hr:j}} Y \iff \widetilde{X}_t \leq_{\text{st:j}} \widetilde{Y}_t, \quad \forall t \textit{ such that } \overline{F}(t,t) > 0.$$

Proof. Let us prove the direct implication. This result follows taking into account inequalities (2.3) and

$$X \leq_{\text{hr:j}} Y \implies X \leq_{\text{st:j}} Y.$$

In order to prove the reverse implication, we consider the characterization of the joint stochastic order in terms of the multivariate stochastic order (see Definition 2.3.c), that is $\widetilde{X}_t \leq_{\text{st:j}} \widetilde{Y}_t$ $\forall t$ such that $\overline{F}(t,t) > 0$, if, and only if,

$$(\widetilde{X}_t, -\widetilde{Y}_t) \leq_{\text{st}} (\widetilde{Y}_t, -\widetilde{X}_t), \quad \forall t \text{ such that } \overline{F}(t,t) > 0.$$

Since the multivariate stochastic order implies the upper orthant order, see Shaked and Shanthinkumar (2007), last inequality implies that

$$P((\widetilde{X}_t, -\widetilde{Y}_t) \in (y,\infty) \times (x,\infty)) \leq P((\widetilde{Y}_t, -\widetilde{X}_t) \in (y,\infty) \times (x,\infty))$$

for all x, y, or equivalently,

$$P(\{X > y, Y < x | X > t, Y > t\}) \leq P(\{Y > y, X < x | X > t, Y > t\}). \tag{2.5}$$

If we consider $x > t$ and $y > t$, and $y \notin S_X \cup S_Y$, where S_X and S_Y denote the supports of X and Y respectively, we can rewrite (2.5) as

$$\begin{aligned} P(\{X > y | X > t, Y > t\}) - P(\{X > y, Y > x | X > t, Y > t\}) \\ \leq P(\{Y > y | X > t, Y > t\}) - P(\{Y > y, X > x | X > t, Y > t\}), \end{aligned}$$

i.e., as

$$\overline{F}(y,t) - \overline{F}(y,x) \leq \overline{F}(t,y) - \overline{F}(x,y),$$

or

$$\overline{F}(x,y) - \overline{F}(y,x) \leq \overline{F}(t,y) - \overline{F}(y,t).$$

Anyway, if $x \in S_X \cup S_Y$, we can get the same equivalence by taking a succession $\{x_n\}_n$ such that $x_n \notin S_X \cup S_Y$, $x_n < x$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. If we consider (2.5) in terms of x_n instead of x, we get

$$\begin{aligned} P(\{X > y | X > t, Y > t\}) - P(\{X > y, Y > x_n | X > t, Y > t\}) \\ \leq P(\{Y > y | X > t, Y > t\}) - P(\{Y > y, X > x_n | X > t, Y > t\}). \end{aligned}$$

for all $n \in \mathbb{N}$. Since the survival function is left continuous, taking limits when $n \to \infty$ we get the equivalence.

Particularly, if we consider $t < x < y$, we get that $\overline{F}(x,y) - \overline{F}(y,x)$ is decreasing in x whenever $x \leq y$ from (2.2), that is equivalent to $X \leq_{\text{hr:j}} Y$. □

The joint weak hazard rate order does not imply, neither is implied by, the joint stochastic order, i.e., neither it is satisfied

$$X \leq_{\text{st:j}} Y \Rightarrow X \leq_{\text{hr:wj}} Y \text{ or } X \geq_{\text{hr:wj}} Y$$

nor

$$X \leq_{\text{hr:wj}} Y \Rightarrow X \leq_{\text{st:j}} Y \text{ or } X \geq_{\text{st:j}} Y.$$

These assertions are proved by the following two counterexamples.

Example 2.1. *Consider the random vector (X,Y) whose probability mass is concentrated at the points $\{P_1,P_2,P_3,P_4\} = \big\{\big(-\frac{4}{5},1\big),\big(\frac{1}{3},\frac{2}{3}\big),\big(\frac{2}{3},\frac{1}{3}\big),\big(1,-\frac{4}{5}\big)\big\}$ as Figure 3 shows. Let us denote*

$$\begin{aligned} p_1 &= P[(X,Y)=P_1] = \frac{1}{3}, \\ p_2 &= P[(X,Y)=P_2] = \frac{1}{6}, \\ p_3 &= P[(X,Y)=P_3] = \frac{1}{3}, \\ p_4 &= P[(X,Y)=P_4] = \frac{1}{6}, \end{aligned}$$

and let $\overline{F}(x,y)$ be their bivariate survival function. It is easy to check that neither $X \leq_{\text{hr:wj}} Y$ nor $X \geq_{\text{hr:wj}} Y$ are satisfied, since

$$\overline{F}(0.2,0.5) = \frac{1}{6} < \frac{2}{6} = \overline{F}(0.5,0.2)$$

and

$$\overline{F}(-1,0.2) = \frac{5}{6} > \frac{4}{6} = \overline{F}(0.2,-1).$$

However, the stochastic inequality $(X,-Y) \leq_{\text{st}} (Y,-X)$ is satisfied, i.e., it holds $X \leq_{\text{st:j}} Y$. To prove it, it is enough to observe that for any upper set U, it holds

$$P[(X,-Y) \in U] \leq P[(Y,-X) \in U]. \tag{2.6}$$

In fact, taking into account the plots of the random vectors $(X,-Y)$ and $(Y,-X)$, which are showed in Figure 4, by the definition of upper sets and the disposition of the points P_i, it is easy to verify that only four different probabilities occur (for each one of the vectors) when the upper set U varies from the upper right corner to the lower left corner:

$$P[(X,-Y) \in U] \in \{p_4, p_3+p_4, p_2+p_3+p_4, 1\} = \left\{\frac{1}{6},\frac{3}{6},\frac{4}{6},1\right\}$$

and

$$P[(Y,-X) \in U] \in \{p_1, p_1+p_2, p_1+p_2+p_3, 1\} = \left\{\frac{2}{6},\frac{3}{6},\frac{5}{6},1\right\}.$$

For example, if the upper set U is the one shown in Figure 4, then $P[(X,-Y) \in U] = 1/6 \leq P[(Y,-X) \in U] = 1/3$. Thus, for all upper sets U the inequality (2.6) holds, i.e., $X \leq_{\text{st:j}} Y$.

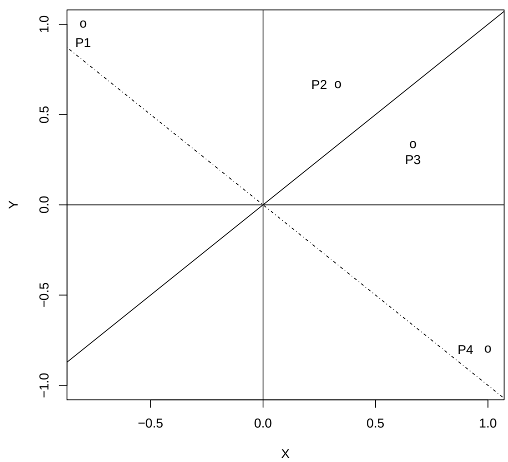

Figure 3: *Plot of the probability density of the vector (X, Y) of Example 2.1.*

Example 2.2. *Now we provide an example of a random vector (X, Y) satisfying $X \geq_{\text{hr:wj}} Y$ but neither $X \leq_{\text{st:j}} Y$ nor $X \geq_{\text{st:j}} Y$. Assume that the probability mass of the random vector (X, Y) is concentrated at the points $\{P_1, P_2, P_3\} = \left\{\left(\frac{1}{3}, 1\right), \left(\frac{2}{3}, \frac{1}{3}\right), \left(1, \frac{2}{3}\right)\right\}$. Like in the previous example, we denote by $p_i = P[(X, Y) = P_i]$ for $i = \{1, 2, 3\}$ and by $\overline{F}(x, y)$ the bivariate survival function of the vector. The regions with different values of the survival function for this random vector are shown in Figure 5. It is easy to check through this figure that it holds $X \geq_{\text{hr:wj}} Y$, that is $\overline{F}(x, y) \leq \overline{F}(y, x)$ for all $x \leq y$, if and only if*

$$\{p_1 \leq p_3, 0 \leq p_3, p_1 \leq p_2\}. \tag{2.7}$$

Also notice that the second one of these conditions fixes the sense of the weak joint hazard order, that is, $X \geq_{\text{hr:wj}} Y$. Thus, let us assume that the conditions in (2.7) are satisfied.

Now, recall that $X \geq_{\text{st:j}} Y$ if and only if $(X, -Y) \geq_{\text{st}} (Y, -X)$, i.e., if the condition (2.6) is satisfied for all upper set U. Thus, in particular, for all $(x, y) \in \mathbb{R}^2$ it should holds

$$P[(X, -Y) \in (x, \infty) \times (y, \infty)] \geq P[(Y, -X) \in (x, \infty) \times (y, \infty)]$$

or equivalently,

$$P[X \geq x, Y \leq y] \geq P[X \leq y, Y \geq x].$$

It is easy to check, through the Figures 6 and 7, that this condition is equivalent to the following set of inequalities:

$$\left\{ \begin{array}{r} p_1 \geq p_2, \\ p_1 \geq p_3, \\ p_1 \geq p_3 + p_2, \\ p_1 \geq 0 \end{array} \right\},$$

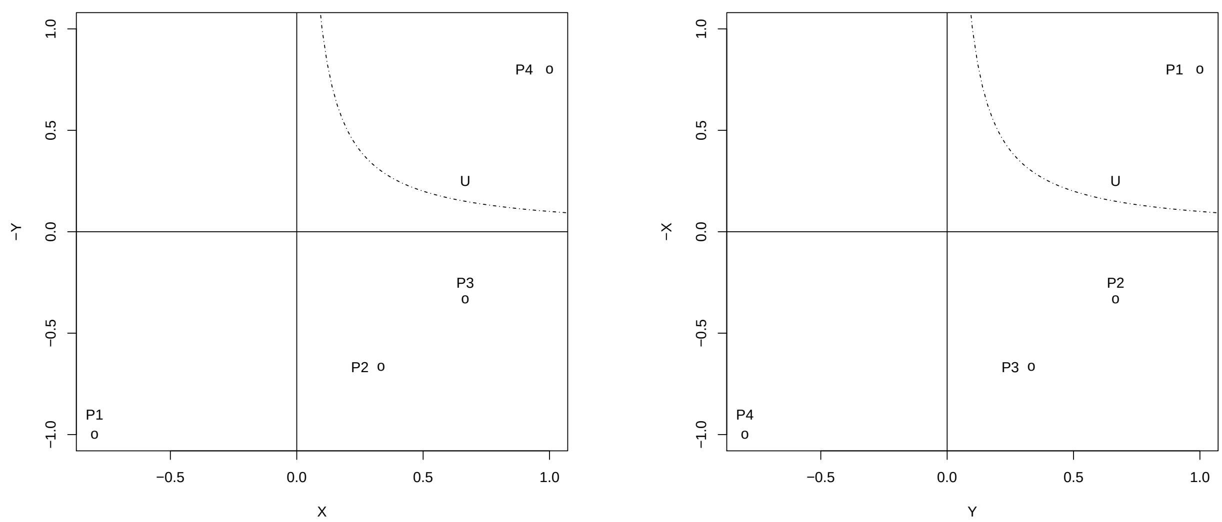

Figure 4: *Plots of the probability densities of the vectors* $(X, -Y)$ *(left side) and* $(Y, -X)$ *(right side) of Example 2.1.*

which is not verified in any sense under (2.7) *for all* p_1, p_2, p_3 *such that* $p_1 + p_2 + p_3 = 1$ *and* $p_i > 0$ *for all* i*. Thus the inequality* $X \geq_{\text{st:j}} Y$ *is not satisfied. In a similar way it can be shown that neither* $X \leq_{\text{st:j}} Y$ *holds.*

Finally we observe that it is easy to see that if $X \leq_{\text{hr:wj}} Y$ then $X \leq_{\text{st}} Y$.

Taking into account these comments and previous results in the literature we have the following chain of implications

$$\begin{array}{ccccc} X \leq_{\text{lr:j}} Y & \Rightarrow & X \leq_{\text{hr:j}} Y & \Rightarrow & X \leq_{\text{hr:wj}} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq_{\text{st:j}} Y & \Rightarrow & X \leq_{\text{st}} Y. \end{array} \tag{2.8}$$

3 Some characterizations and properties

In this section we provide some results for the new order and applications of these results.

First of all, we give a characterization of the new joint order in terms of the expectations of a particular class of functions. This result is similar to several results for the joint orders considered by Shanthikumar and Yao (1991). Along the proof given a function $g \in D$ we define $\Delta g(x, y) = g(x, y) - g(y, x)$.

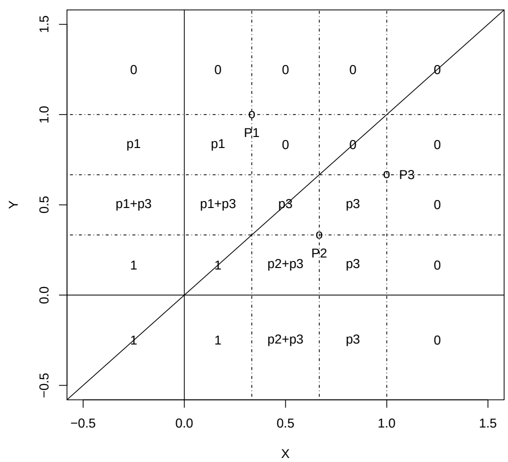

Figure 5: *Plot of* $\overline{F}(x,y)$.

Theorem 3.1. *Let us consider the set*

$$G_{\rm hr}^{w} = \{g \in D : \lim_{x \to -\infty} \Delta g(x,y) = 0 \textit{ and } \forall\ v \leq v',$$
$$\Delta g(u,v') - \Delta g(u,v) \textit{ is increasing in } u,\ u \leq v\}.$$

Then $X \leq_{\rm hr:wj} Y$ *if, and only if,* $E[g(X,Y)] \geq E[g(Y,X)]$ *for all* $g \in G_{\rm hr}^{w}$*, provided previous expectations exist.*

Proof. First we prove the direct implication. Following the proof of Theorem 3.17 in Shanthinkumar and Yao (1991) and assuming the differentiability of $g(u,v)$ we get

$$E[\Delta g(X,Y)] = \int_{\mathbb{R}} \int_{u}^{\infty} \frac{\partial}{\partial v} \Delta g(u,v) \big(\frac{\partial}{\partial v} \overline{F}(v,u) - \frac{\partial}{\partial v} \overline{F}(u,v) \big) dv du.$$

Now changing the order of integration, we ge that

$$\begin{aligned} E[\Delta g(X,Y)] &= \int_{\mathbb{R}} \int_{-\infty}^{v} \frac{\partial}{\partial v} \Delta g(u,v) \big(\frac{\partial}{\partial v} \overline{F}(v,u) - \frac{\partial}{\partial v} \overline{F}(u,v) \big) du dv \\ &= \int_{\mathbb{R}} \int_{-\infty}^{v} \frac{\partial^2}{\partial u \partial v} \Delta g(u,v) \big(\overline{F}(u,v) - \overline{F}(v,u) \big) du dv \geq 0, \end{aligned}$$

where the last equality follows integrating by parts and recalling that, by assumption, $\lim_{u \to -\infty} \Delta g(u,v) = 0$. Last inequality follows from the fact that $X \leq_{\rm hr:wj} Y$ and the properties of $\Delta g(u,v)$.

For the reverse implication, we take $g(u,v) = \mathbf{1}\{u \geq x, v \geq y\}$ for $x \leq y$, so that $\Delta g(u,v) = \mathbf{1}\{u \geq x, x > v \geq y\}$. It is easy to see that $g(u,v) \in G_{\rm hr}^{w}$. Therefore, for all $x \leq y$,

$$E[\Delta g(X,Y)] = \overline{F}(x,y) - \overline{F}(y,x) \geq 0.$$

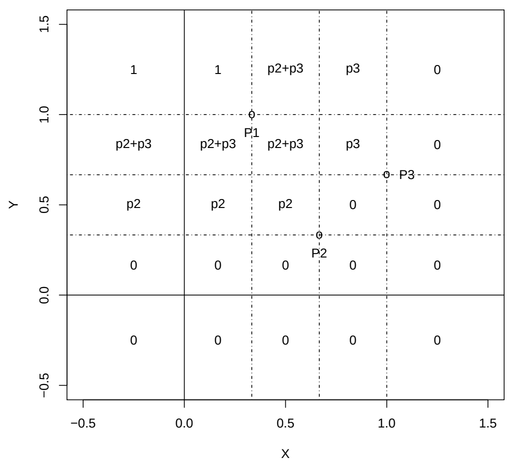

Figure 6: *Plot of $P[X \geq x, Y \leq y]$ in Example 2.2.*

□

Remark 3.1. *We notice that, in particular, we can assume $\lim_{x\to-\infty} \Delta g(x,y) = 0$ for all random variables with finite left point in their support.*

We also observe that if we take $g \in G^w_{\mathrm{hr}}$, since $\Delta g(u,v') - \Delta g(u,v)$ starts equal to zero when $u \to -\infty$ and it is increasing in u, then we have $\Delta g(u,v') \geq \Delta g(u,v)$ for all $u \leq v \leq v'$, that means, $g \in G_{\mathrm{hr}}$. Therefore $G^w_{\mathrm{hr}} \subseteq G_{\mathrm{hr}}$.

To finish we want to mention that the condition $\Delta g(u,v') - \Delta g(u,v)$ increasing in u, with $u \leq v$, $\forall v \leq v'$, is equivalent to the supermodularity of $\Delta g(u,v)$ on the set $u \leq v$ (see Shaked and Shanthikumar, 2007, p. 335, for the definition of supermodular functions and related properties).

Next we introduce an example where we can apply the previous theorem.

Example 3.1. *In portfolio selection problems a common way to compare alternatives is by means of expected utilities: given two portfolios Z_1 and Z_2, a risk averse agent prefers Z_1 to Z_2 if $E[u(Z_1)] \geq E[u(Z_1)]$, where the function U represents its utility function u, usually assumed to be increasing and concave (see, e.g., Kijima and Ohnishi, 1996). In particular, logarithmic utility functions are often considered, i.e., utility functions are assumed to be defined as $u(z) = \ln(1+z)$. Consider now a pair (X,Y) of non independent and non-negative random returns such that $X \leq_{\mathrm{hr:wj}} Y$. Then consider the portfolios $Z_1 = \alpha Y + (1-\alpha)X$ and $Z_2 = (1-\alpha)Y + \alpha X$, with $\alpha \in [0,1]$, and assume one wants to choose among the two portfolios according to the logarithmic expected utility criteria. For it, define $g(x,y) = \ln\big(\alpha y + (1-\alpha)x\big)$, so that $\Delta g(x,y) = \ln\big(\alpha y + (1-\alpha)x\big) - \ln\big(\alpha x + (1-\alpha)y\big)$. It is easy to verify that $\Delta g(x,y)$ is supermodular on the set $x \leq y$ whenever $\alpha \geq \frac{1}{2}$. Thus, by Theorem*

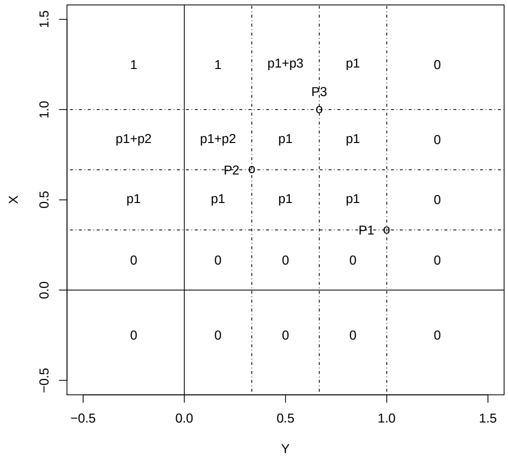

Figure 7: *Plot of $P[X \leq y, Y \geq x]$ in Example 2.2.*

3.1 one has $E[g(X,Y)] \geq E[g(Y,X)]$, i.e., portfolio Z_1 is preferred to portfolio Z_2 according to the above mentioned criteria.

Next we prove that, under appropriate assumptions, the hr:wj and hr:j orders are equivalent. The following technical lemma we will used to prove this fact.

Lemma 3.1. *Let us consider a bivariate random vector (X,Y) with joint survival function $\overline{F}$ and joint density function f. If $X \leq_{\text{hr:wj}} Y$ then*

$$\int_y^\infty f(y,z)dz \geq \int_y^\infty f(z,y)dz \quad \textit{for all } y \geq 0.$$

Proof. First we observe that

$$\begin{aligned} X \leq_{\text{hr:wj}} Y \quad &\Longleftrightarrow \quad \overline{F}(x,y) \geq \overline{F}(y,x) \;\text{ for all } x \leq y \\ &\Longleftrightarrow \quad \overline{F}(x,y) - \overline{F}(y,y) \geq \overline{F}(y,x) - \overline{F}(y,y) \;\text{ for all } x \leq y. \end{aligned}$$

Therefore

$$\begin{aligned} X \leq_{\text{hr:wj}} Y \quad &\Longrightarrow \quad \lim_{x \to y^-} \frac{\left(\overline{F}(y,x) - \overline{F}(y,y)\right)}{y-x} \geq \lim_{x \to y^-} \frac{\left(\overline{F}(x,y) - \overline{F}(y,y)\right)}{y-x} \text{ for all } x \geq 0 \\ &\Longleftrightarrow \quad \int_y^\infty f(y,z)dz \geq \int_y^\infty f(z,y)dz \text{ for all } y \geq 0 \end{aligned}$$

□

The following result provides the key to prove the equivalence among the hr:wj and hr:j orders under additional conditions.

Proposition 3.1. *Let (X, Y) a random vector with joint density function f such that, for all x in the union of the supports of X and Y, the function $h_x(z) = f(x, z) - f(z, x)$ has at most one sign change, from negative to positive in $[x, \infty)$, when it occurs. If*

$$\int_x^\infty f(x, z)dz \geq \int_x^\infty f(z, x)dz \text{ for all } x \geq 0, \tag{3.1}$$

then $X \leq_{\text{hr:j}} Y$.

Proof. Let

$$h(x, y) = \int_y^\infty \big(f(x, z) - f(z, x)\big)dz.$$

Since $\frac{\partial}{\partial y}h(x, y) = -f(x, y) + f(y, x)$, then $h(x, y)$ is non increasing or it has a maximum in $[x, \infty)$, by hypothesis. Moreover, $\lim_{y\to\infty} h(x, y) = 0$ and $\lim_{y\to x^+} h(x, y) \geq 0$, where last inequality follows from (3.1). Then $h(x, y) \geq 0$ for all $x \leq y$, or equivalently, $X \leq_{\text{hr:j}} Y$. $\square$

Notice that under the crossing condition among $f(x, y)$ and $f(y, x)$, which appears in last proposition, the weak joint hazard rate order implies the strong one, that is,

$$X \leq_{\text{hr:wj}} Y \Rightarrow X \leq_{\text{hr:j}} Y,$$

since $X \leq_{\text{hr:wj}} Y$ implies the condition (3.1) by Lemma 3.1. Therefore we can establish the following theorem.

Theorem 3.2. *Let (X, Y) a random vector with joint density function f such that, for all x in the union of the supports of X and Y, the function $h_x(z) = f(x, z) - f(z, x)$ has at most one sign change, from negative to positive in $[x, \infty)$, when it occurs. Then*

$$X \leq_{\text{hr:wj}} Y \Leftrightarrow X \leq_{\text{hr:j}} Y.$$

In the following it is shown an example of a random vector (X, Y) satisfying the assumptions of Proposition 3.1, but such that the joint density function is not arrangement increasing or decreasing, and therefore such that X and Y are not ordered in the joint likelihood ratio order.

Example 3.2. *Let (X, Y) be a random vector with bivariate normal distribution, that is, with density function*

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}}\, e^{-\frac{1}{2(1-p^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2p(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)} \quad x, y \in \mathbb{R}.$$

Let us consider a fixed $x \in \mathbb{R}$, then consider the condition $f(x, y) - f(y, x) = 0$ or, equivalently, the condition $h_x(y) = \log f(x, y) - \log f(y, x) = 0$. Since $h_x(y)$ is a parabola as a function of y for a fixed $x \in \mathbb{R}$, $h_x(y)$ has at most two zeros, where one of them occurs at

$y = x$. Thus $h_x(y)$ has at most one sign change on (x, ∞). If the extreme point of $h_x(y)$ is a minimum or, equivalently, if $\sigma_1 < \sigma_2$, then the sign sequence will be the required in the Theorem 3.1, that is, $-, +$. Moreover, we get that

$$h_x(y) = \frac{1}{2(1-p^2)} \left(k_1 y^2 + 2k_2 y - k_1 x^2 - 2k_2 x\right),$$

where $k_1 = \sigma_1^{-2} - \sigma_2^{-2}$ and $k_2 = \mu_2\sigma_2^{-2} - \mu_1\sigma_1^{-2} + p(\mu_2 - \mu_1)\sigma_1^{-1}\sigma_2^{-1}$. Therefore, the roots of this equation are $y_1 = x$ and $y_2 = -2k_2/k_1 - x$. To get a sign change on (x, ∞) it should be $y_2 > x$, which is equivalent to $-k_2/k_1 > x$. Since $-k_2/k_1$ is a finite real number, there will be a point $x_0 = -k_2/k_1$ such that $h_x(y)$ has a sign change on the interval (x, ∞) for all $x < x_0$, whenever $u_X < -k_2/k_1$ (where u_X denote the left extreme point of the support of X). This means that it does not hold neither $X \leq_{\mathrm{lr:j}} Y$ nor $X \geq_{\mathrm{lr:j}} Y$. This situation it is showed in Figure 8 for $(X, Y) \sim N\left((1, 0.5), \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}\right)$ by taking $x = 0$ to show an example of the crossing condition.

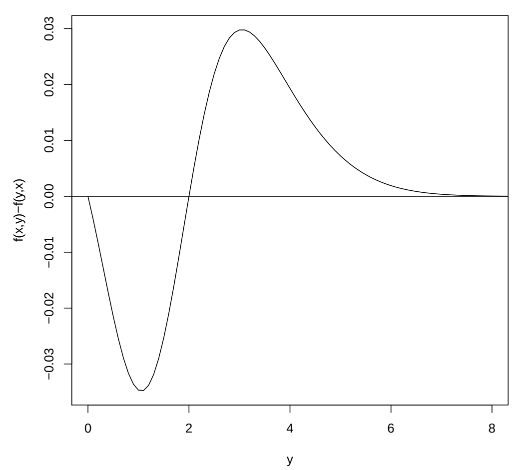

Figure 8: *Plot of $h_0(y)$.*

To sum up, whenever it is satisfies that $\sigma_1 < \sigma_2$ and $u_X < -k_2/k_1$, then the crossing condition of Theorem 3.1 holds, and therefore it does not hold $X \leq_{\mathrm{lr:j}} Y$ nor $X \geq_{\mathrm{lr:j}} Y$, while, if it occurs the opposite (i.e., $u_X > -k_2/k_1$), then it holds $X \leq_{\mathrm{lr:j}} Y$ when $\sigma_1 < \sigma_2$ and $X \geq_{\mathrm{lr:j}} Y$ otherwise. However, if $\sigma_1 < \sigma_2$ and $u_X < -k_2/k_1$ it can happen that $X \geq_{\mathrm{hr:j}} Y$. In fact, it can be numerically verified with R package that, for the same random vector as in Figure 8, the condition (3.1) *holds, as Figure 9 shows. Then by Proposition 3.1 we can conclude $X \leq_{\mathrm{hr:j}} Y$.*

Remark 3.2. *Notice that in Theorem 2.15 in Shanthinkumar and Yao (1991) the condition $u_X < -k_2/k_1$ is not satisfied, being $u_X = 0$ and $-k_2/k_1 < 0$. Therefore there is not an x*

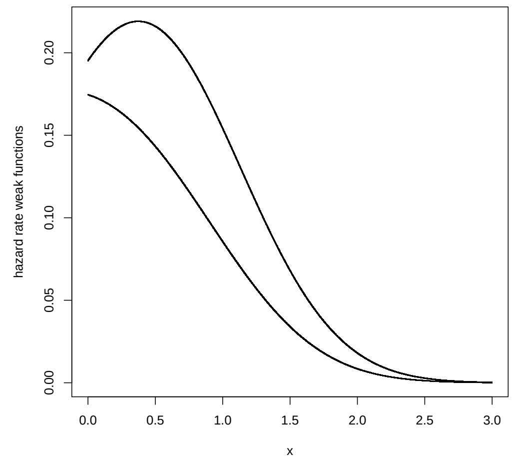

Figure 9: *Plot of* $\int_x^\infty f(x,z)dz$ *and* $\int_x^\infty f(z,x)dz$.

such that $h_x(y) = f(x,y) - f(y,x)$ *has a sign change and the likelihood joint order holds. Then Example 3.2 improves the Theorem 2.15 in Shanthinkumar and Yao (1991), since it gives a completed study of all situations where the likelihood joint order holds for two random variables with normal bivariate joint density function.*

The $\leq_{\mathrm{hr:wj}}$ does not imply, nor is implied, by the $\leq_{\mathrm{hr}}$ order, as shown in the introductory examples, unless the random variables are independent. However, under appropriate conditions on the dependence between X and Y, it is possible to verify that $\leq_{\mathrm{hr:wj}}$ implies the standard $\leq_{\mathrm{hr}}$ order, or viceversa, as shown in the next theorems. In these statements, to describe the dependence structure in the vector (X,Y) we make use of the notion of copula (actually, of survival copula) that we recall here.

Given a bivariate vector (X,Y) with joint distribution F and marginal distributions F_X and F_Y, a function $C_{(X,Y)} : [0,1] \to \mathbb{R}_+$ is said to be the copula of (X,Y) if $F(x,y) = C_{(X,Y)}(F_X(x), F_Y(y))$, for all $(x,y) \in \mathbb{R}^2$. It is well known that if the marginal distributions are continuous then the copula $C_{(X,Y)}$ is unique, and that copulas entirely describe the dependence structure of (X,Y). In some applications, dealing with lifetimes of individuals or components, it is commonly considered the survival copula instead of the copula. Let $\overline{F}$ be the joint survival function of (X,Y), then a function $\overline{C}_{(X,Y)} : [0;1] \to \mathbb{R}_+$ is said to be the survival copula of (X,Y) if $\overline{F}(x,y) = \overline{C}_{(X,Y)}(\overline{F}_X(x), \overline{F}_Y(y))$, for all $(x,y) \in \mathbb{R}^2$. We refer the readers to the monograph Nelsen (2006) for further details.

In particular, for the next two statements we need a property of bivariate copulas which was firstly introduced in Bassan and Spizzichino (2005), and further studied and applied in dependence analysis in Durante and Ghiselli Ricci (2009 and 2012), where it is called *supermigrativity.*

Definition 3.1. *A bivariate copula $C : [0,1]^2 \longrightarrow [0,1]$ is called supermigrative (submigrative) if it is symmetric, i.e. $C(u,v) = C(v,u)$ for every $(u,v) \in [0,1]^2$, and it satisfies*

$$C(u, \gamma v) \geq (\leq)\; C(\gamma u, v) \tag{3.2}$$

for all $u \leq v$ and $\gamma \in (0,1)$. We say that C is strictly supermigrative (submigrative) if the inequality in (3.2) is strict whenever $u \leq v$ and $\gamma \in (0,1)$.

We can now state the conditions for implications between the standard hazard order and the weak joint hazard order.

Theorem 3.3. *Let the survival copula $\overline{C}_{(X,Y)}$ of (X,Y) be supermigrative. Then $X \leq_{\mathrm{hr}} Y$ implies $X \leq_{\mathrm{hr:wj}} Y$.*

Proof. Fix any $x, y \in \mathbb{R}$ such that $x \leq y$. The assumption $X \leq_{\mathrm{hr}} Y$ implies both

$$\overline{F}_X(x)\overline{F}_Y(y) \geq \overline{F}_Y(x)\overline{F}_X(y) \tag{3.3}$$

and

$$\overline{F}_X(y) \leq \overline{F}_Y(y) \; \forall y \in \mathbb{R}. \tag{3.4}$$

Observe that, from (3.4) and $x \leq y$, it follows

$$\overline{F}_X(y) \leq \overline{F}_Y(y) \leq \overline{F}_Y(x) \quad \text{and} \quad \overline{F}_X(y) \leq \overline{F}_X(x) \leq \overline{F}_Y(x), \tag{3.5}$$

while nothing can be asserted on the order between $\overline{F}_Y(y)$ and $\overline{F}_X(x)$, being $\overline{F}_X(x) \leq \overline{F}_Y(y)$ and $\overline{F}_X(x) > \overline{F}_Y(y)$ both possible.

Assume that $\overline{F}_X(x) \leq \overline{F}_Y(y)$, i.e., let

$$\overline{F}_X(y) \leq \overline{F}_X(x) \leq \overline{F}_Y(y) \leq \overline{F}_Y(x). \tag{3.6}$$

Define $\overline{F}_X(y) = u_1$, $\overline{F}_Y(y) = v_1$, $\overline{F}_Y(x) = v_2$ and $u_2 = \dfrac{\overline{F}_X(y) \cdot \overline{F}_Y(x)}{\overline{F}_Y(y)}$. Note that $u_2 \leq \overline{F}_X(x)$ because of inequality (3.3). Thus

$$\begin{aligned}
\overline{F}(x,y) &= \overline{C}_{(X,Y)}\big(\overline{F}_X(x), \overline{F}_Y(y)\big) \\
&= \overline{C}_{(X,Y)}\big(\overline{F}_X(x), v_1\big) \\
&\geq \overline{C}_{(X,Y)}\big(u_2, v_1\big) \\
&\geq \overline{C}_{(X,Y)}\big(u_1, v_2\big) \\
&= \overline{C}_{(X,Y)}\big(\overline{F}_X(y), \overline{F}_Y(x)\big) = \overline{F}(y,x),
\end{aligned}$$

where the first inequality follows from monotonicity of $\overline{C}_{(X,Y)}$, and the second from the hypothesis of supermigrativity of the copula (which is equivalent to $\overline{C}(u_2, v_1) \geq \overline{C}(u_1, v_2)$ for all $u_1 \leq u_2 \leq v_1 \leq v_2$ such that $u_1 \cdot v_2 = u_2 \cdot v_1$).

Assume now that $\overline{F}_X(x) > \overline{F}_Y(y)$, i.e., let

$$\overline{F}_X(y) \leq \overline{F}_Y(y) < \overline{F}_X(x) \leq \overline{F}_Y(x). \tag{3.7}$$

In this case define $\overline{F}_X(y) = u_1,\ \overline{F}_X(x) = v_1$ and $\overline{F}_Y(x) = v_2$, and note that $u_2 = \dfrac{u_1 \cdot v_2}{v_1} \leq \overline{F}_Y(y)$. Thus again

$$\begin{aligned}
\overline{F}(x,y) &= \overline{C}_{(X,Y)}\big(\overline{F}_X(x), \overline{F}_Y(y)\big) \\
&\geq \overline{C}_{(X,Y)}\big(\overline{F}_X(x), u_2\big) \\
&= \overline{C}_{(X,Y)}\big(v_1, u_2\big) \\
&= \overline{C}_{(X,Y)}\big(u_2, v_1\big) \\
&\geq \overline{C}_{(X,Y)}\big(u_1, v_2\big) \\
&= \overline{C}_{(X,Y)}\big(\overline{F}_X(y), \overline{F}_Y(x)\big) = \overline{F}(y,x),
\end{aligned}$$

where the equality $\overline{C}_{(X,Y)}\big(v_1, u_2\big) = \overline{C}_{(X,Y)}\big(u_2, v_1\big)$ follows from the exchangeability of the copula. □

Next statement provides conditions for the implication in the opposite direction.

Theorem 3.4. *Let the survival copula $\overline{C}_{(X,Y)}$ of (X,Y) be submigrative. Then $X \leq_{\mathrm{hr:wj}} Y$ implies $X \leq_{\mathrm{hr}} Y$.*

Proof. Recall that the assumption $X \leq_{\mathrm{hr:wj}} Y$ implies $X \leq_{\mathrm{st}} Y$, i.e.,

$$\overline{F}_X(t) \leq \overline{F}_Y(t) \quad \forall t \in \mathbb{R}. \tag{3.8}$$

From (3.8) and for all $x \leq y \in \mathbb{R}$, it is possible to have

$$\overline{F}_X(y) \leq \overline{F}_X(x) \leq \overline{F}_Y(y) \leq \overline{F}_Y(x), \tag{3.9}$$

or

$$\overline{F}_X(y) \leq \overline{F}_Y(y) \leq \overline{F}_X(x) \leq \overline{F}_Y(x). \tag{3.10}$$

On the other hand, the assumption $X \leq_{\mathrm{hr:wj}} Y$ is equivalent to

$$\overline{F}(y,x) \leq \overline{F}(x,y) \quad \forall\ x \leq y \in \mathbb{R}$$

that is,

$$\overline{C}(\overline{F}_X(y), \overline{F}_Y(x)) \leq \overline{C}(\overline{F}_X(x), \overline{F}_Y(y)). \tag{3.11}$$

Note that, as in the previous theorem, the condition of submigrativity on the copula can be restated as

$$\overline{C}(u_2, v_1) \leq \overline{C}(u_1, v_2) \text{ for all } u_1 \leq u_2 \leq v_1 \leq v_2 \text{ such that } u_1 \cdot v_2 = u_2 \cdot v_1. \tag{3.12}$$

Fix $x < y \in \mathbb{R}$. Assume (3.9) and let $\overline{F}_X(y) = u_1,\ \overline{F}_X(x) = u_2,\ \overline{F}_Y(y) = v_1,\ \overline{F}_Y(x) = v_2$, Thus from (3.11) we obtain

$$\overline{C}(u_1, v_2) \leq \overline{C}(u_2, v_1) \quad \text{for} \quad u_1 \leq u_2 \leq v_1 \leq v_2, \tag{3.13}$$

but (3.13) contradicts the assumption on the copula if $u_1 \cdot v_2 = u_2 \cdot v_1$. Therefore, $u_1 \cdot v_2 <$ (or $>$) $u_2 \cdot v_1$, i.e.

$$\frac{u_1}{v_1} < (\text{or } >) \frac{u_2}{v_2} \Longleftrightarrow \frac{\overline{F}_X(y)}{\overline{F}_Y(y)} < (\text{or } >) \frac{\overline{F}_X(x)}{\overline{F}_Y(x)}. \tag{3.14}$$

Observe that the inequality in (3.14) stand up for all $x < y \in \mathbb{R}$ in the same direction, because equality can never hold (otherwise there is a contradiction) and because of continuity of $\overline{F}_X$ and $\overline{F}_Y$. Thus it holds $X \leq_{\text{hr}}$ (or $\geq_{\text{hr}}$) Y. Since $X \leq_{\text{st}} Y$, clearly $X \geq_{\text{hr}} Y$ cannot be true, thus $X \leq_{\text{hr}} Y$ holds.

For the case (3.10), if we assign $\overline{F}_X(y) = u_1,\ \overline{F}_Y(y) = u_2,\ \overline{F}_X(x) = v_1,\ \overline{F}_Y(x) = v_2$, then similar discussion would be enough to prove $X \leq_{\text{hr}} Y$. □

Examples of copulas satisfying the condition in the previous theorems can be provided considering the results provided in Bassan and Spizzichino (2005). For it, recall that a copula C is called Archimedean if it can be written in the form

$$C(u, v) = \phi^{-1}(\phi(u) + \phi(v)), \tag{3.15}$$

for any continuous and strictly decreasing function $\phi : [0, 1] \mapsto [0, \infty]$ such that $\phi(1) = 0$. In this case, the function ϕ is called generator of the Archimedean copula. Bassan and Spizzichino (2005), Proposition 6.1, proved that any Archimedean semi-copula with the generator function ϕ is supermigrative (submigrative) if and only if ϕ^{-1} is log-convex (log-concave). Thus, since any copula is a semi-copula, an Archimedean copula is supermigrative (submigrative) if and only if its generator is log-convex (log-concave). By making use of this property it is easy to verify that the well known Clayton copulas and Gumbel-Hougaard copulas satisfy the supermigrativity property, while the Gumbel-Barnett copulas satisfy the submigrativity property.

Another example of copulas satisfying the supermigrativity (submigrativity) property is provided here.

Example 3.3. *Let (X, Y) have a Farlie-Gumbel-Morgenstern (FGM) survival copula, that is, let*

$$\overline{C}(u, v) = uv(1 + \theta(1 - u)(1 - v)) \quad \text{with } \theta \in [-1, 1].$$

Let $0 \le \ u_1 \le u_2 \le v_1 \le v_2 \le 1$ be such that $u_1 \cdot v_2 = u_2 \cdot v_1$, or equivalently $\frac{v_1}{v_2} = \frac{u_1}{u_2}$; then

$$\begin{aligned}
u_2 \le v_2 \iff & (v_1 - v_2)u_2 \ge (v_1 - v_2)v_2 \\
\iff & \frac{v_1 - v_2}{v_2} \ge \frac{v_1 - v_2}{u_2} \\
\iff & \frac{v_1}{v_2} - 1 \ge \frac{v_1}{u_2} - \frac{v_2}{u_2} \\
\iff & \frac{v_1}{v_2} + \frac{v_2}{u_2} \ge \frac{v_1}{u_2} + 1 \\
\iff & \frac{u_1}{u_2} + \frac{v_2}{u_2} \ge \frac{v_1}{u_2} + 1 \\
\iff & u_1 + v_2 \ge v_1 + u_2 \\
\iff & 1 - u_1 - v_2 + u_1 v_2 \le 1 - v_1 - u_2 + v_1 u_2 \\
\iff & (1 - u_1)(1 - v_2) \le (1 - u_2)(1 - v_1).
\end{aligned}$$

Thus, from the last inequality, and by using again the equality $u_1 \cdot v_2 = u_2 \cdot v_1$, we obtain

$$u_1 v_2 (1 + \theta(1 - u_1)(1 - v_2)) \le \ (\ge) \ \ u_2 v_1 (1 + \theta(1 - u_2)(1 - v_1))$$

for all $\theta > \ (<) \ 0$ that is, $\overline{C}(u_1, v_2) \le \ (\ge) \ \overline{C}(u_2, v_1)$. Being the FGM copula a symmetric copula, the assumptions of the previous statements are satisfied.

The following statement is the closure under mixture of the $\le_{\text{hr:wj}}$. It should be pointed out that this property is not satisfied by the standard hazard rate order.

Proposition 3.2. *Let Θ be any random variable assuming values in $\mathfrak{T}$, and let (X, Y) be defined as the mixture of the family $\{(X, Y)_\theta, \theta \in \mathfrak{T}\}$ with respect to Θ (i.e., $\overline{F}(x, y) = \int_{\mathfrak{T}} \overline{F}_\theta(x, y) dP[\Theta = \theta]$) If $X_\theta \le_{\text{hr:wj}} Y_\theta$ for all $\theta \in \mathfrak{T}$, then $X \le_{\text{hr:wj}} Y$.*

Proof. Since $X_\theta \le_{\text{hr:wj}} Y_\theta$, then

$$\overline{F}_\theta(x, y) \ge \overline{F}_\theta(y, x) \ \text{ for all } \ x \le y.$$

Thus, for all $x \le y$

$$\overline{F}(x, y) = \int_{\mathfrak{T}} \overline{F}_\theta(x, y) dP[\Theta = \theta] \ \ge \ \int_{\mathfrak{T}} \overline{F}_\theta(y, x) dP[\Theta = \theta] = \overline{F}(y, x),$$

that is, $X \le_{\text{hr:wj}} Y$. □

From previous result we can easily provide examples of models such that the inequality $X \le_{\text{hr:wj}} Y$ is satisfied. Let us consider some examples. The first one arises in the context of frailty models.

Example 3.4. *Let (X,Y) have joint survival function defined as*

$$\overline{F}(x,y) = \int_{\mathfrak{T}} [\overline{F}_1(x)\overline{F}_2(y)]^\theta dP[\Theta = \theta].$$

If $\overline{F}_1 \leq_{\mathrm{hr}} \overline{F}_2$ then $X \leq_{\mathrm{hr:wj}} Y$.

Next we describe a similar situation for the case of accelerated life models.

Example 3.5. *Let us consider two units with random lifetimes S and T. Suppose that the units are working in a common operating environment, which is represented by a random variable Θ, independent of S and T and having effect on the units in the form*

$$X = \frac{S}{\Theta} \quad \text{and} \quad Y = \frac{T}{\Theta}.$$

If Θ has support on $(1,+\infty)$ then the components are working in a harsh environment, while if it has support on $(0,1)$ then the components are working in a gentler environment (see Ma, 1999).

In this case the joint survival function of (X,Y) is given by

$$\overline{F}(x,y) = \int_{\mathbb{R}} \overline{F}(x\theta)\overline{G}(y\theta) dP[\Theta = \theta],$$

where $\overline{F}$ and $\overline{G}$ are the survival functions of T and S respectively. It is clear that if $S \leq_{\mathrm{hr}} T$ then $X \leq_{\mathrm{hr:wj}} Y$.

4 Application to a data set from a crossover study

In this section we discuss the new notion for a data set in medical research coming from a crossover study in particular. Crossover trials is an important class of statical methods used in medical research to evaluate the effect of some pharmaceutical treatment. In contrast to the classical parallel design, in which experimental units (patients) are randomized to a treatment and remain on that treatment throughout the duration of the trial, a crossover trial is a repeated measurements design such that each patient receives different treatments during different time periods, i.e., such that the patients randomly cross over from one treatment to another during the course of the trial.

Even if crossover trials are generally restricted to the study of short term outcomes in chronic diseases, because the disease needs to persist long enough for the investigator to expose the subject to each of the experimental treatments and measure the response, the treatment effects can be estimated with greater precision. In fact, moving patient variation according to crossover trials makes them potentially more efficient than similar sized, parallel group trials in which each subject is exposed to only one treatment.

The data set considered here comes from the study described in Senn et al. (1997), and can be downloaded at `http://www.senns.demon.co.uk/Data/Selipati.xls`. In such a study, a number of patients suffering from asthma have been subjected to different doses of two formulations of formoterol (a potent drug acting on asthma) and a placebo in random order. After each treatment, an index describing the log area under the curve for forced expiratory volume in one second (shortly, log-AUC) has been recorded. The data considered in our analysis are the log-AUC after the placebo treatment and after the ISF24 treatment (a particular dose of a formulation of formoterol); thus, two measures for each patient have been collected and compared. Figure 10 is the scatterplot of the data set after the removal of missing data.

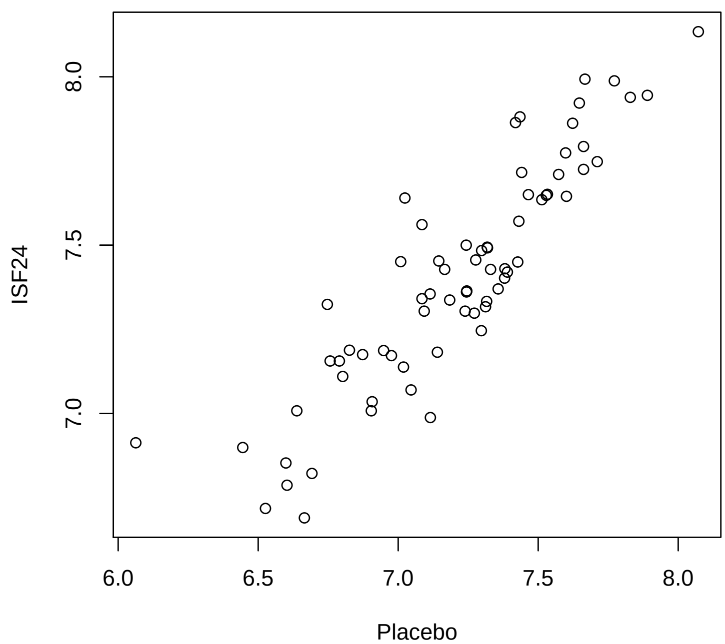

Figure 10: *Scatterplot of the log-AUC measures under placebo and under ISF24 treatments.*

A comparison of the marginal distributions of the log-AUC index under ISF24 treatment and placebo treatment has been performed, showing that the log-AUC after ISF24 treatment is stochastically greater than the log-AUC after placebo treatment (see Figure 11).

In Figure 12 we have a graphical representation of the P-P plot for the data set. The hazard rate order can be rewritten in terms of a property of the P-P plot. A subset A of the Euclidean space is called star-shaped with respect to a point s if for every $x \in A$ we have that A contains the whole line segment between x and s. A real function f is called star-shaped with respect to a point (a, b), if its epigraph is star-shaped with respect to (a, b). Now if we consider the P-P plot, we have that the hazard rate holds if, and only if, the P

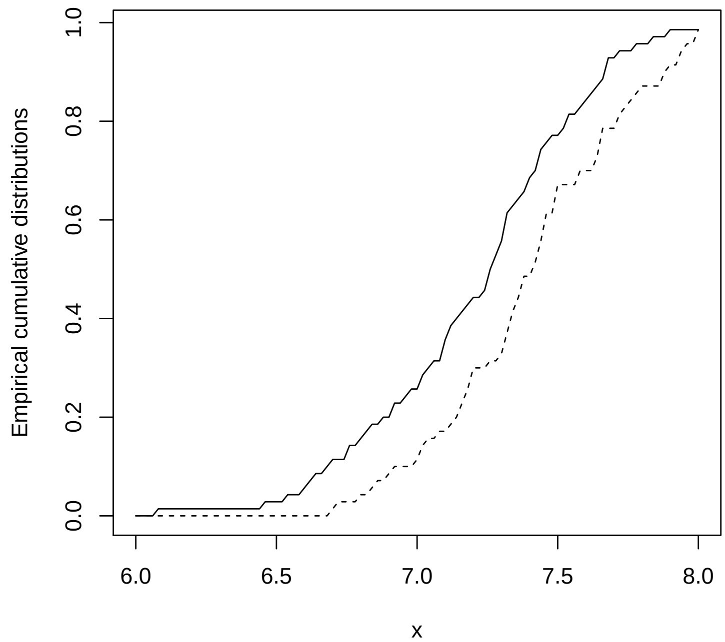

Figure 11: *Empirical cumulates of the log-AUC measures under placebo (continuous line) and under ISF24 (dotted line) treatments.*

- P plot is star-shaped with respect to $(1, 1)$ (see Müller and Stoyan, 2002, p.9). This is the case for the empirical P-P plot, thus we can assert that the hazard rate order holds between the log-AUC after ISF24 treatment and the log-AUC after placebo treatment.

However, this comparison is just a comparison based on marginal distributions, thus not suitable to affirm that the ISF24 treatment improves the expiratory volume of each patient, i.e., it does not guarantee that forced expiratory volume in one second of each patient is greater under ISF24 treatment than under placebo trial. For it, a joint comparison is needed. Thus, an analysis of the copula of the pair of log-AUC under the two trials has been performed. A scatterplot of the copula is shown in Figure 13. From this scatterplot it is possible to recognize a Gumbel-Hougaard copula, and a goodness of fit test for this copula, performed with the `copula` package of `R`, does not reject (with p-value equal to 0.6698) the hypothesis of a Gumbel-Hougaard copula with parameter $\theta = 3.602$.

Thus, by Theorem 3.3, we can assert that the weak joint hazard order holds, i.e., that the log-AUC under ISF24 treatment is stochastically greater than the log-AUC under placebo treatment whatever is the minimal value they can assume.

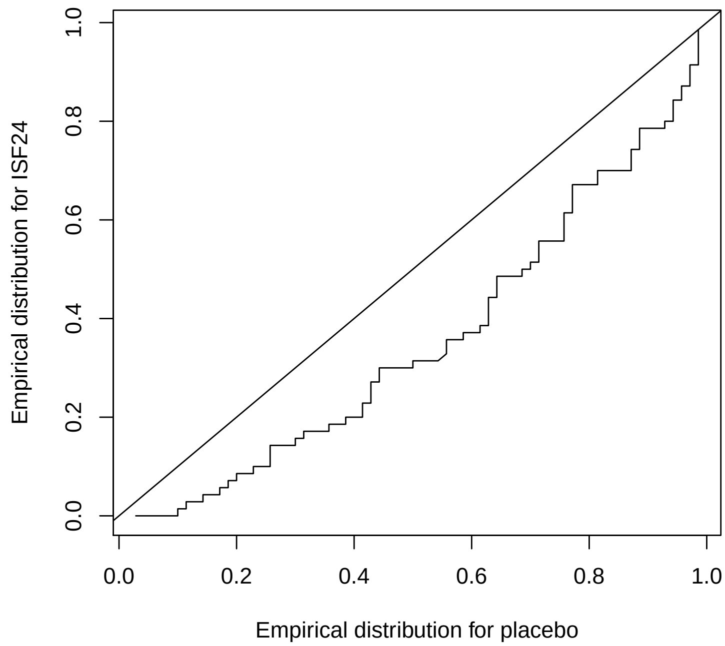

Figure 12: *P-P plot of the empirical distribution functions of the log-AUC measures under placebo and ISF24 treatments.*

5 Conclusions and further remarks

In this paper we have considered a new notion for the comparison of random variables, taking into account the dependence among the two random variables. This notion had appeared, implicitly, in some other papers or as a dependence notion. Its applications can be found in the literature and new applications can be given in several contexts as can be seen in sections 3 and 4.

It is interesting to notice that this notion can be extended to the general case of n not necessarily independent components. This new notion parallels some other multivariate generalizations provided in Shanthikumar and Yao (1991) and Belzunce, Martínz-Puertas and Ruiz (2013). Next we give such generalization.

Definition 5.1. *Given random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *with joint survival function* $\overline{F}$*, we say that* $X_1 \leq_{\text{hr:wj}} X_2 \leq_{\text{hr:wj}} \ldots \leq_{\text{hr:wj}} X_n$*, if*

$$[X_i - t \mid X_k > t; k \in \{1, 2, ..., n\}] \leq_{\text{st}} [X_j - t \mid X_k > t; k \in \{1, 2, ..., n\}]$$

for all t*, such that* $\overline{F}(t, \ldots, t)$*, and for every* $i < j$*.*

It is easy to verify that $X_1 \leq_{\text{hr:wj}} X_2 \leq_{\text{hr:wj}} \ldots \leq_{\text{hr:wj}} X_n$ if, and only if, $\overline{F}(x_1, \ldots, x_n)$ is

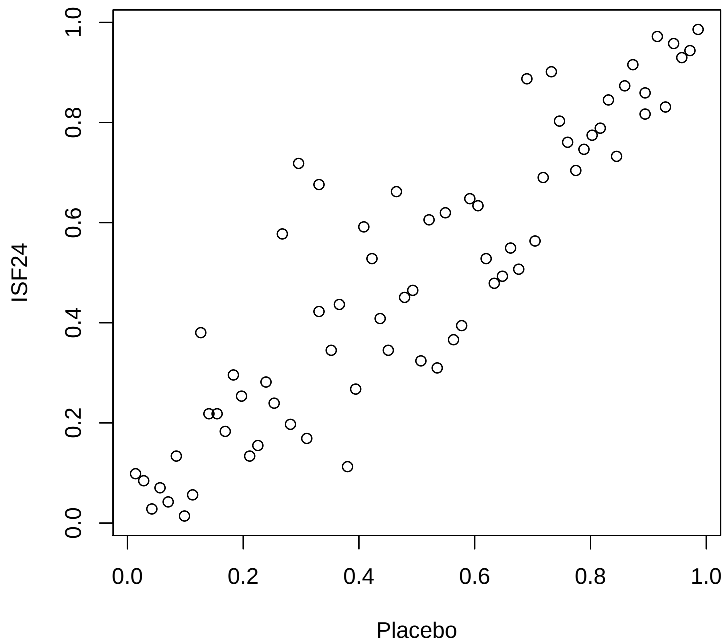

Figure 13: *Empirical copula of the log-AUC measures under placebo and ISF24 treatments.*

arrangement decreasing, and therefore is equivalent to the UOAI notion considered by Cai and Wei (2014). Further results can be provided, in a similar way to the bivariate case, for this new multivariate notion.

To finish we want to point out that a similar notion in the bivariate and multivariate case, can be given replacing the joint survival function by the joint distribution function. This notion leads to a "joint weak reversed hazard rate" order. For this notion can be carried out a similar study, that closely parallels the results given in this paper.

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