Finite-size corrections to disordered systems on Erdös-Rényi random graphs

U. Ferrari,[1] C. Lucibello,[2] F. Morone,[2] G. Parisi,[3] F. Ricci-Tersenghi,[3] and T. Rizzo[4]

[1]*Laboratoire de Physique Théorique de l'ENS, CNRS and UPMC, 24 rue Lhomond, F-75005 Paris, France*

[2]*Dipartimento di Fisica, Università "La Sapienza," Piazzale A. Moro 2, I-00185, Rome, Italy*

[3]*Dipartimento di Fisica, INFN–Sezione di Roma 1, CNR-IPCF UOS Roma Kerberos, Università "La Sapienza," Piazzale A. Moro 2, I-00185, Rome, Italy*

[4]*CNR-IPCF, UOS Roma Kerberos, Dipartimento di Fisica, Università "La Sapienza," Piazzale A. Moro 2, I-00185, Rome, Italy*

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We study the finite-size corrections to the free-energy density in disordered spin systems on sparse random graphs, using both replica theory and the cavity method. We derive analytical expressions for the $O(1/N)$ corrections in the replica symmetric phase as a linear combination of the free energies of open and closed chains. We perform a numerical check of the formulas on the random-field Ising model at zero temperature by computing finite-size corrections to the ground-state energy density.

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I. INTRODUCTION

The critical behavior of ferromagnets in the presence of a random magnetic field is not well understood in spite of the great efforts made in the past. Dimensional reduction (i.e., the critical exponents of this system in D dimensions are the same as those of a pure ferromagnet in $d = D - 2$ dimensions) is perturbatively correct, but it fails beyond perturbation theory. However, it is not clear at the present moment if dimensional reduction is a valid approximation in some range of dimensions and what the form of deviations from dimensional reduction is. Different scenarios have been presented in the literature, and they will not be discussed here: we aim to construct a new approach to the problem.

The difficulties are related to the following facts: (1) The phase transition is dominated by the zero-temperature fixed point: the critical exponents as a function of temperature are the same as those as a function of the magnetic field at zero temperature.[1] (2) The supersymmetric scenario (dimensional reduction) assumes the essential uniqueness of the solution of the local mean-field equations $m_i = \tanh(\beta h_i^{\text{eff}})$, where $h_i^{\text{eff}} \equiv \sum_k J_{ik} m_i + h_i$, which become $m_i = \text{sign}(h_i^{\text{eff}})$ at zero temperature.

The crux of the supersymmetry argument is that already at temperatures higher than the critical temperature and certainly at zero temperature, the mean-field equations have multiple solutions.[2]

These observations imply it would be wise to use a field-theoretic approach directly at zero temperature, perturbing around a mean-field model where multiple solutions of the mean-field equations are present. Unfortunately, this is not straightforward to do. The perturbation theory is usually constructed as an expansion around the mean-field theory, and the preferred mean-field theory is the one for the infinite-range model.

In the infinite-range model in the infinite-volume limit the solution of the mean-field equations is essentially unique (apart from a time-reversal symmetry),[3] and we cannot perform any expansion around a nonexistent transition with multiple solutions. However, we must not throw out the baby with the bath water. This disappointing situation disappears on the Erdös-Rényi (ER) and other sparse random graphs, where the coordination number is finite and a more complex mean-field theory is valid, where an exponential number of solutions are present (we may have many different solutions for the same value of the global magnetization[4]).

The locality of the model on ER graphs, where the properties of a spin depend on the local magnetization averaged over its finite neighborhood, makes this problem deeply different from the infinite-range model, where only the global magnetization is relevant. Therefore we believe that studying finite-dimensional models performing an expansion around the ER model is a mandatory investigation that may hold some surprises.

Our long-term goal is to construct a new perturbation expansion around the ER graph results along the lines discussed in some previous works.[5–7] The construction of such a loop expansion for finite-dimensional models is a rather complex task. In this paper we present a first step in this direction, i.e., the study of the $1/N$ correction around the mean-field solution for the ER graph. The tools that we use in this computation are the same of those that we should use in finite dimensions. Independent of this long-term goal, the study of finite N corrections is an interesting well-studied problem because these corrections usually tell us something about the nature of the phase, and the appearance of divergence in these corrections is often a signal of incorrectness of the mean-field construction.

In the domain of physical spin systems, diluted models represent a class of mean-field-like systems sharing an essential feature of the finite-dimensional ones, which is the finite coordination number. As a consequence, diluted models should mimic the physics of real systems better than the fully connected ones (we have already remarked that this is what happens for zero-temperature ferromagnets in random magnetic fields). Moreover, when dealing with finite systems, the peculiar structure of diluted networks should give a first insight into how the topology can modify thermodynamic quantities. Indeed, diluted models are defined on random graphs which are locally treelike and have typical loops of size $O(\log N)$. However, for finite (and small) sizes these loops become short and much more similar to the short loops which are abundant in any finite-dimensional network (think, e.g., of lattice models). In this sense we can interpret the $1/N$

corrections in diluted models as a way to expand towards finite-dimensional models.

Finite-size corrections to the free energy have been investigated in fully connected systems,[8–10] mean-field optimization problems,[11,12] and some simple disorder systems,[13] sometimes as a by-product of the Hessian diagonalization.[14] However, to our knowledge, only a solution in zero external field has been derived for sparse random graphs[15,16] in the replica symmetric phase. In the following we will use the replica method in order to compute disorder-averaged corrections to the free energy. An obvious limitation of the method is that is cannot be applied on a given realization of the disorder to obtain corrections to the estimates provided by the belief propagation (BP) algorithm, which corresponds to the Bethe approximation. In order to tackle this problem a sequence of algorithms of increasing computational complexity was proposed in Ref. 15, and it was later shown that they indeed reduce systematically the error on the BP estimates.[17] The sole limitation of these algorithms is that they do not give corrections to the free energy, only to local observables, notably the energy and the magnetization.

This paper is organized as follows. In Sec. II we define the model. In Sec. III we compute finite-size corrections of the free-energy density in finitely connected models, using the replica formalism. In Sec. IV we make the same calculation using the cavity method. Since the cavity method is well defined only in the thermodynamic limit, it has to be reinvented in order to handle finite-size systems. We find that both procedures (replica and cavity) give the same expression for the $1/N$ free-energy density corrections, and in this respect, they are completely equivalent also beyond the thermodynamic limit. The cavity method allows a more precise physical interpretation of the finite-size corrections and of their connections with highly correlated topological structures (loops in the random graph). In Sec. V we test our analytical predictions by performing a numerical experiment on the zero-temperature random-field Ising model and computing the $1/N$ corrections to the ground-state energy. Numerical results are found to be in excellent agreement with the analytical prediction.

II. THE MODEL

We consider a model of N interacting Ising spins $\{\sigma_i = \pm 1\}_{i=1}^{N}$ defined by the following Hamiltonian:

$$\mathcal{H} = -\sum_{i<j} C_{ij}\, J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \tag{1}$$

where we have decoupled the topology of the underlying graph, encoded in the symmetric adjacency matrix $\{C_{ij}\}$, from the exchange interactions $\{J_{ij}\}$. The numbers C_{ij} specify the particular graph considered and take values $C_{ij} = 1$ or 0 depending on whether sites i and j are connected or not. Here we consider Erdös-Rényi random graphs,[18] which can be generated by sampling the adjacency matrix from the following distribution:[19]

$$\mathcal{P}(\{C_{ij}\}) = \prod_{i<j}\left[\frac{z}{N}\delta(C_{ij}-1) + \left(1-\frac{z}{N}\right)\delta(C_{ij})\right]. \tag{2}$$

The spins interact among each other via quenched random couplings J_{ij}, which are assumed to be identically independently distributed (or fixed to a single value J). Moreover we allow the spins to interact with a local magnetic field (random or nonrandom). The disorder-averaged free-energy density of the system, at the temperature $T = \beta^{-1}$, is defined as

$$\begin{aligned} f(\beta,N) &= -(\beta N)^{-1}[\log Z_N(\beta)]_{\rm av} \\ &= f_0(\beta) + \frac{1}{N}f_1(\beta) + o\left(\frac{1}{N}\right), \end{aligned} \tag{3}$$

where the average has to be performed over the topological disorder and the quenched randomness. The main part of this work is devoted to the analytical computation of the $f_1(\beta)$ term, the finite-size correction to the free energy. The calculation can be performed in two different ways, known as the replica method and the cavity method. The latter derivation is particularly useful in order to better understand the physical meaning of the results, which is less clear in the replica picture.

III. COMPUTING THE FREE-ENERGY DENSITY WITH REPLICAS

The replica calculation of the free-energy density starts from the following well-known identity:

$$[\log Z_N(\beta)]_{\rm av} = \lim_{n\to 0}\frac{\partial}{\partial n}\log[(Z_N(\beta))^n]_{\rm av}. \tag{4}$$

The moments of the partition function $[(Z_N(\beta))^n]_{\rm av}$ are then evaluated for integer values of number of replicas n. At the end of the calculation, the analytical continuation to real values of n allows us to take the limit $n \to 0$. The replicated averaged partition function reads (from now on we drop the dependence of Z_N on β)

$$\begin{aligned} [(Z_N)^n]_{\rm av} = \Bigg[&{\rm Tr}\left(\prod_{i<j}\exp\left(\beta J_{ij}C_{ij}\sum_a^n \sigma_i^a\sigma_j^a\right)\prod_i \right. \\ &\left.\times \exp\left(\beta h_i\sum_a^n\sigma_i^a\right)\right)\Bigg]_{\rm av}. \end{aligned} \tag{5}$$

Performing the average over the topological disorder using the distribution (2) and setting

$$\begin{aligned} V(\sigma,\tau) &\equiv N\log\left\{1+\frac{z}{N}\left[\overline{\exp\left(\beta J\sum_a\sigma^a\tau^a\right)}^{J}-1\right]\right\}, \\ B(\sigma) &\equiv \log\left[\overline{\exp\left(\beta h\sum_a\sigma^a\right)}^{h}\right] - \frac{1}{2N}V(\sigma,\sigma), \end{aligned} \tag{6}$$

Eq. (5) takes the following form:

$$[(Z_N)^n]_{\rm av} = {\rm Tr}\left[\exp\left\{\frac{1}{2N}\sum_{i,j}V(\sigma_i,\sigma_j) + \sum_i B(\sigma_i)\right\}\right]. \tag{7}$$

We can achieve the site factorization of Eq. (7) by means of the order parameter

$$\tilde{\rho}(\sigma) = N^{-1}\sum_i\prod_a\delta\left(\sigma^a-\sigma_i^a\right). \tag{8}$$

Enforcing Eq. (8) with a δ functional in Eq. (7), we trace over the decoupled sites and then integrate out $\tilde{\rho}(\sigma)$ which appears

in a Gaussian form. We arrive at an expression for the fields $\rho(\sigma)$ suitable for saddle-point evaluation:[20]

$$[(Z_N)^n]_{\rm av} = \sqrt{\det(V)} \int [D\rho] e^{-N S[\rho]}. \tag{9}$$

The replicated action $S[\rho]$ is given by

$$S[\rho] = \frac{1}{2} \int d\sigma d\tau \; \rho(\sigma)\, V(\sigma,\tau) \rho(\tau) - \log \int d\sigma \times \exp\left[\int d\tau \; V(\sigma,\tau)\rho(\tau) + B(\sigma) \right], \tag{10}$$

where the symbol $\int d\sigma$ is a proxy for the more cumbersome notation $\int d\sigma \equiv \prod_{a=1}^{n} \sum_{\sigma^a = \pm 1}$. Let us now extract the leading-order contribution in the replicated action $S[\rho]$. We define the matrix $U(\sigma,\tau)$ and the vector $H(\sigma)$ from the first-order expansion in N of Eq. (6) to be

$$U(\sigma,\tau) \equiv \overline{\exp\left(\beta J \sum_a \sigma^a \tau^a \right)}^{J}, \qquad H(\sigma) \equiv \log \left[\overline{\exp\left(\beta h \sum_a \sigma^a \right)}^{h} \right], \tag{11}$$

and we write the thermodynamically relevant part of the action (10) as $S[\rho] = S_0[\rho] + o(1)$, where

$$S_0[\rho] = \frac{z}{2} \int d\sigma d\tau \; \rho(\sigma)\, [U(\sigma,\tau) - 1] \rho(\tau) - \log \int d\sigma \times \exp\left\{ z \int d\tau \; [U(\sigma,\tau) - 1]\rho(\tau) + H(\sigma) \right\}. \tag{12}$$

The leading-order free energy f_0 comes from the saddle point of Eq. (12), followed by the limit $n \to 0$, as we will see in the next section. A first $O(\frac{1}{N})$ correction to the free energy comes from the $O(\frac{1}{N})$ term in Eq. (10) evaluated at the saddle point.

A. Leading free energy

We now evaluate the functional integral (9) by the steepest descent method:

$$\lim_{N \to +\infty} -\frac{1}{N} \log[(Z_N)^n]_{\rm av} = S_0[\rho_*], \tag{13}$$

where $\rho_*(\sigma)$ is the solution of the saddle-point equation:

$$\frac{\delta S_0[\rho]}{\delta \rho(\sigma)} = 0 \longrightarrow \rho_*(\sigma) = \frac{\exp\left[z \int d\,\sigma' U(\sigma,\sigma') \rho_*(\sigma') + H(\sigma) \right]}{\int d\sigma \exp\left[z \int d\sigma' U(\sigma,\sigma') \rho_*(\sigma') + H(\sigma) \right]}. \tag{14}$$

In order to take the small-n limit we have to use an appropriate parametrization for the order parameter $\rho_*(\sigma)$. If we assume a replica-symmetric (RS) ansatz, a convenient parametrization for $\rho_*(\sigma)$ is given by

$$\rho_*(\sigma) = \int dh\, P(h) \left[\frac{\exp(\beta h \sum_a \sigma^a)}{[2\cosh(\beta h)]^n} \right]. \tag{15}$$

Inserting this parametrization in Eq. (14) and taking the limit $n \to 0$, we obtain the usual self-consistent cavity equations for the distributions $P(h)$ and $Q(u)$ of cavity fields and bias, respectively:

$$P(h) = \sum_{k=0}^{\infty} \frac{z^k}{k!} e^{-z} \overline{\int \left[\prod_{i=1}^{k} dQ(u_i) \right] \delta\left(h - h_R - \sum_{i=1}^{k} u_i \right)}^{h_R}, \qquad Q(u) = \overline{\int dP(h)\, \delta\left[u - \frac{1}{\beta} \tanh^{-1}[\tanh(\beta J)\tanh(\beta h)] \right]}^{J}. \tag{16}$$

The RS free-energy density can then be estimated as

$$f_0(\beta) = \beta^{-1} \lim_{n \to 0} \frac{\partial}{\partial n} S_0[\rho_*] \tag{17}$$

and can be explicitly written in terms of the distributions $P(h)$ and $Q(u)$.[21]

B. Fluctuations around the RS saddle point

The Gaussian integral obtained by expanding Eq. (10) around the saddle point generates the order $1/N$ corrections. We set

$$\rho(\sigma) = \rho_*(\sigma) + \frac{\chi(\sigma)}{\sqrt{N}}, \quad S^{(2)}(\sigma,\sigma';\rho) = \frac{\delta^2 S_0[\rho]}{\delta\rho(\sigma)\delta\rho(\sigma')}. \tag{18}$$

Expanding the action in powers of $1/N$, we find

$$S[\rho] = S_0[\rho_*] + \frac{1}{N} S_1[\rho_*] + \frac{1}{2N} \int d\sigma d\sigma' \chi(\sigma) S^{(2)}(\sigma,\sigma';\rho_*) \chi(\sigma') + o(N^{-1}), \tag{19}$$

where $S_1[\rho_*]$ is given by the following expression:

$$S_1[\rho_*] = \frac{z}{2} \int d\sigma [U(\sigma,\sigma) - 1]\rho_*(\sigma) + \frac{z^2}{4} \int d\sigma d\sigma' \rho_*(\sigma)[U(\sigma,\sigma') - 1]^2 \rho_*(\sigma'). \tag{20}$$

The functional integral (9) at this order evaluates

$$\begin{aligned} -\frac{1}{N} \log[(Z_N)^n]_{\rm av} &= S_0[\rho_*] + \frac{1}{N} S_1[\rho_*] + \frac{1}{2N} \log\det(1 - T) + o(N^{-1}) \\ &= S_0[\rho_*] + \frac{1}{N} S_1[\rho_*] - \frac{1}{2N} \sum_{L=1}^{\infty} \frac{{\rm Tr}[T^L]}{L} + o(N^{-1}), \end{aligned} \tag{21}$$

where the matrix $T(\sigma,\sigma')$ reads

$$T(\sigma,\sigma') = z \left[U(\sigma,\sigma')\rho_*(\sigma') - \left(\int d\tau U(\sigma,\tau)\rho_*(\tau) \right) \rho_*(\sigma') \right]. \tag{22}$$

Using the RS parametrization (15), it turns out that in the limit $n \downarrow 0$ the trace ${\rm Tr}(T^L)$ can be arranged in a linear combination of free energies of closed and open chains. It all comes down

to the fact that the term $U(\sigma,\sigma')\rho_*(\sigma')$, present in $T(\sigma,\sigma')$, can be linked to the replicated transfer matrix of an edge receiving a cavity field at one of its extremities. In Appendix B we prove the following formula:

$$\frac{\partial}{\partial n}\mathrm{Tr}(T^L) = -\beta z^L\left[\phi_L^c - L\left(\phi_L^a - \phi_{L-1}^a\right)\right] + O(n), \quad (23)$$

where $\phi_L^{c/a}$ are free energies of closed and open spin chains in a graph of length $L \geqslant 1$, with ϕ_0^a defined as $\phi_0^a \equiv -\beta^{-1}\mathbb{E}_h \log 2\cosh(\beta h)$. Writing the RS free-energy density as

$$f_{\mathrm{RS}} = f_{\mathrm{RS}}^{(0)} + \frac{1}{N} f_{\mathrm{RS}}^{(1)} + o(N^{-1}) \quad (24)$$

and observing that the term $S_1[\rho_*]/N$ [namely, Eq. (20)] cancels out with part of the first two terms in the sum $\sum_{L=1}^{\infty}\mathrm{Tr}(T^L)/(2NL)$, the finite-size correction of the RS free-energy density $f_{\mathrm{RS}}^{(1)}$ can be evaluated as

$$f_{\mathrm{RS}}^{(1)} = \left(z - \frac{z^2}{2}\right)\phi_0^a - \frac{z}{2}\phi_1^a - \frac{z^2}{2}\left(\phi_2^a - 2\phi_1^a\right) + \frac{1}{2}\sum_{L=3}^{\infty}\frac{z^L}{L}\left[\phi_L^c - L\left(\phi_L^a - \phi_{L-1}^a\right)\right]. \quad (25)$$

The sum entering the previous formula can be considered as a sum over independent loops weighted with the factor $[\phi_L^c - L(\phi_L^a - \phi_{L-1}^a)]$ by noticing that, in the thermodynamic limit, $z^L/(2L)$ is exactly the average number of loops of length L in an Erdös-Rényi random graph of mean connectivity z. The same formula holds true on the Erdös-Rényi ensemble $\mathbb{G}(N,M)$, where $M = zN/2$ is the fixed number of edges, since the distribution of topological structures such as the number of finite loops remains the same at the $1/N$ order.

In the limit of vanishing external field, Eq. (25), evaluated in the paramagnetic phase, takes the following simpler form:

$$f_{\mathrm{RS}}^{(1)} = \frac{z}{2\beta}\mathbb{E}_J \log\cosh(\beta J) - \frac{1}{2\beta}\sum_{L=3}^{\infty}\frac{z^L}{L}\mathbb{E}_{\{J_i\}}\log\left[1 + \prod_{i=1}^{L}\tanh(\beta J_i)\right], \quad (26)$$

where the first term takes into account the fact that the average number of links is $z(N-1)/2$ and the second one is the contribution of all loops of length $L \geqslant 3$. The loops we are talking about are topologically defined as non-self-intersecting closed paths. Self-intersecting closed paths would give contributions proportional to N^{-2} since the self-intersection is observed, on average, in a fraction N^{-2} of the total number of vertices. While Eq. (25) is an original contribution to the literature, its zero-field counterpart Eq. (26) has already been presented.[15] Moreover the full distribution of $f^{(1)}$ in the absence of external field and in the RS phase has been rigorously computed,[16] and it is consistent with the mean value given by Eq. (26).

IV. COMPUTING THE FREE-ENERGY DENSITY WITH THE CAVITY METHOD

We now show how to compute the finite-size corrections to the free-energy density using the cavity method. The reason to be interested in such a calculation is twofold. First, we have to corroborate the physical insight gained from replicas; second, we want to establish the equivalence of the two methods beyond the leading order, showing how both procedures also give the same result at order $1/N$.

The cavity method is well defined only in thermodynamic limit. In order to study $1/N$ corrections to the free-energy density of a model defined on an Erdös-Rényi random graph (ERRG), we need to define a new ensemble of random graphs of $\mathcal{N}$ vertices, such that in the limit $\mathcal{N} \to \infty$ any topological structure appears with the same density it has in the ERRG of N vertices. Here we are assuming that the free energy of a model of N variables can be written as $F_N = Nf(\{d_i\})$, where $f(\{d_i\})$ is the free-energy density computed in the thermodynamic limit of a model having the same *densities* d_i of topological structures appearing in the finite N model. The new ensemble we are going to define is required to compute such a free-energy density.

The topological structures we are interested in are the only ones that give contributions up to order $O(\frac{1}{N})$, i.e., linear chains of length L (i.e., with L edges and $L+1$ vertices) and loops of length L. Let us start by computing their densities in an ERRG of N sites, where each link is present with probability z/N. The density of linear chains of size L (i.e., the number of linear chains per node) is

$$d_L^{\mathrm{chain}} = \frac{1}{N}\left(\frac{z}{N}\right)^L \frac{1}{2}N(N-1)\cdots(N-L) \simeq \frac{z^L}{2}\left(1 - \frac{L(L+1)}{2N}\right), \quad (27)$$

and the density of loops of length L is

$$d_L^{\mathrm{loop}} = \frac{1}{N}\left(\frac{z}{N}\right)^L \frac{1}{2L}N(N-1)\cdots(N-L+1) \simeq \frac{1}{N}\frac{z^L}{2L}. \quad (28)$$

In the new ensemble a random graph of $\mathcal{N}$ nodes can be viewed as the union of basic topological structures (BTS), which, for the present purposes, are chains and loops. The graph can be built in the following way. For each $L \geqslant 1$, consider all sequences of $L+1$ different indices $(i_0,i_1,\ldots,i_L)$ with the condition $i_0 < i_L$, which avoids double counting a chain; for each sequence of indices draw the edges $(i_0,i_1),(i_1,i_2),\ldots,(i_{L-1},i_L)$ with probability $a_L/\mathcal{N}^L$. Then, for each $L \geqslant 3$, consider all sequences of L different indices $(i_1,i_2,\ldots,i_L)$ with the conditions that i_1 is the smallest among the L indices and $i_2 < i_L$ (these two conditions ensure that each loop is counted only once); for each sequence of indices draw the edges $(i_1,i_2),(i_2,i_3),\ldots,(i_{L-1},i_L),(i_L,i_1)$ with probability $c_L/\mathcal{N}^{L-1}$.

A useful representation of this graph is in terms of a factor graph, where the variable nodes are the graph nodes and the factor nodes are the BTS. Thanks to the scaling of the probabilities used in building the graph, the corresponding factor graph is sparse since the total number of BTS (i.e., of

factor nodes) is given by

$$\sum_{L=1}^{\infty} \frac{\mathcal{N}(\mathcal{N}-1)\cdots(\mathcal{N}-L)}{2} \frac{a_L}{\mathcal{N}^L} + \sum_{L=3}^{\infty} \frac{\mathcal{N}(\mathcal{N}-1)\cdots(\mathcal{N}-L+1)}{2L} \frac{c_L}{\mathcal{N}^{L-1}} \simeq \mathcal{N}\left(\sum_{L=1}^{\infty}\frac{a_L}{2} + \sum_{L=3}^{\infty}\frac{c_L}{2L}\right),$$

and coefficients a_L and c_L are constants.

The sparsity of the factor graph ensures that the whole construction is consistent in the $\mathcal{N}\to\infty$ limit. Indeed, the probability that any pair of graph nodes enters in more than one BTS is $O(1/\mathcal{N})$. Since in the new ensemble we are interested in computing the free-energy density to leading order, we can safely assume that any two graph nodes interact through at most one BTS; this BTS uniquely determines whether the edge between the two graph nodes is present or not.

The factor graph representation also allows us to write down the free-energy density in a standard way by summing factor-node and variable-node contributions,

$$f = \frac{1}{2}\sum_{k=1}^{\infty} a_k \phi_k^a + \frac{1}{2}\sum_{k=3}^{\infty}\frac{c_k}{k}\phi_k^c + \phi_{\text{site}}, \tag{29}$$

where ϕ_k^a and ϕ_k^c are, respectively, the free energies of chains and loops of length k and

$$\phi_{\text{site}} = \frac{T}{\mathcal{N}}\sum_i (1-n_i)\sum_{\sigma_i}\mu_i(\sigma_i)\log\mu_i(\sigma_i)\,,$$

with $\mu_i(\sigma_i)$ being the single-spin marginal and n_i being the number of BTS where the variable i enters.

We should now determine the values of coefficients a_k and c_k such that the densities of chains and loops in a typical graph of the new ensemble match those in Eqs. (27) and (28) in the large-$\mathcal{N}$ limit. When computing the actual density of a given topological structure (e.g., a chain or a loop), one should consider that such a topological structure can coincide with a BTS or be part of a BTS or involve more than one BTS.

As a warm-up, let us compute the density of links (chains of length $L=1$) in the limit $\mathcal{N}\to\infty$:

$$d_1^{\text{chain}} = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\frac{\mathcal{N}^2}{2}\left[\sum_{k=1}^{\infty} k\mathcal{N}^{k-1}\frac{a_k}{\mathcal{N}^k} + \sum_{k=3}^{\infty}\mathcal{N}^{k-2}\frac{c_k}{\mathcal{N}^{k-1}}\right] = \frac{1}{2}\left[\sum_{k=1}^{\infty} k a_k + \sum_{k=3}^{\infty} c_k\right], \tag{30}$$

where $k\mathcal{N}^{k-1}$ in the first sum and $\mathcal{N}^{k-2}$ in the second sum are, respectively, the number of chains and loops of length k passing through a given link, i.e., the number of possible BTS containing the two variables connected by a given link.

When computing the density of topological structures made of more than one link, we need to consider that such structures can overlap with more than one BTS. In order to be concrete let us consider the density of chains of length $L=2$:

$$d_2^{\text{chain}} = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\frac{\mathcal{N}^3}{2}\left[\left(\frac{2d_1^{\text{chain}}}{\mathcal{N}}\right)^2 + \sum_{k=2}^{\infty}(k-1)\mathcal{N}^{k-2}\frac{a_k}{\mathcal{N}^k} + \sum_{k=3}^{\infty}\mathcal{N}^{k-3}\frac{c_k}{\mathcal{N}^{k-1}}\right], \tag{31}$$

where $2d_1^{\text{chain}}/\mathcal{N} \equiv p_1$ is the probability of having a link.[22] The general expression for densities of linear chains of length $L\geqslant 3$ is the following:

$$d_L^{\text{chain}} = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\frac{\mathcal{N}^{L+1}}{2}\left[S_L\left(\frac{2d_1^{\text{chain}}}{\mathcal{N}},\ldots,\frac{2d_{L-1}^{\text{chain}}}{\mathcal{N}^{L-1}}\right) + \sum_{k=L}^{\infty}(k-L+1)\frac{a_k}{\mathcal{N}^L} + \sum_{k=L+1}^{\infty}\frac{c_k}{\mathcal{N}^L}\right], \tag{32}$$

where the function S_L gives the probability that the L consecutive links come from more than one BTS and can be written (see Appendix A) in terms of the probabilities of having $k(<L)$ consecutive links: $p_k \equiv 2d_k^{\text{chain}}/\mathcal{N}^k$. Since each term in function S_L is of order $\mathcal{N}^{-L}$, in the limit $\mathcal{N}\to\infty$ we have

$$\begin{aligned} 2d_L^{\text{chain}} &= \mathcal{N}^L S_L(p_1,\ldots,p_{L-1}) + \sum_{k=L}^{\infty}(k-L+1)a_k + \sum_{k=L+1}^{\infty} c_k \\ &= S_L\big(2d_1^{\text{chain}},\ldots,2d_{L-1}^{\text{chain}}\big) + \sum_{k=L}^{\infty}(k-L+1)a_k \\ &\quad + \sum_{k=L+1}^{\infty} c_k = z^L\left(1-\frac{L(L+1)}{2N}\right). \end{aligned} \tag{33}$$

Note that Eq. (33) is valid for any $L\geqslant 1$ since $S_1\equiv 0$ and $c_2\equiv 0$.

A similar expression can be written for the densities of loops of length L:

$$d_L^{\text{loop}} = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\frac{\mathcal{N}^L}{2L}\left[R_L(p_1,\ldots,p_{L-1}) + \frac{c_L}{\mathcal{N}^{L-1}}\right], \tag{34}$$

where again the function R_L represents the probability of generating a loop of size L by more than one BTS. Since the probability of having k consecutive links is $O(\mathcal{N}^{-k})$, the function R_L is $O(\mathcal{N}^{-L})$ and then

$$d_L^{\text{loop}} = \frac{c_L}{2L} = \frac{1}{N}\frac{z^L}{2L} \implies c_L = \frac{z^L}{N} \quad \text{for} \quad L\geqslant 3. \tag{35}$$

In other words, making a loop by randomly choosing smaller structures is more improbable than directly randomly generating such a loop.

The detailed computation of coefficients a_k from Eq. (33) is given in Appendix A. Here we just quote the result:

$$a_1 = z + \frac{1}{N}(2z^2 - z), \tag{36}$$

$$a_L = \frac{1}{N}(z^{L+1} - z^L) \quad \text{for} \quad L\geqslant 2. \tag{37}$$

Plugging these coefficients in Eq. (29), we finally get

$$f = \frac{z}{2}\left(1-\frac{1}{N}\right)\phi_1^a + \frac{z^2}{2N}\left(2\phi_1^a - \phi_2^a\right) + \frac{1}{N}\sum_{L=3}^{\infty}\frac{z^L}{2L}\left[\phi_L^c - L\left(\phi_L^a - \phi_{L-1}^a\right)\right] + \phi_{\text{site}}. \quad (38)$$

We observe that the sum on the right-hand side matches the sum over loops entering Eq. (25). Moreover, if the term ϕ_{site} is expressed by means of cavity fields, one finds exactly Eq. (25). This can be immediately seen in the case of zero external field in all the paramagnetic phases, where variables are unbiased and we have $\phi_{\text{site}} = -T(1-\ell)\log 2$, with $\ell = z + (z^2/2 - z)/N$ being the density of edges in the factor graph (i.e., the number of edges in the factor graph per variable node). Substituting this expression for ϕ_{site} in Eq. (38), simply gives

$$f = -T\left(\log 2 - \frac{z}{2}\mathbb{E}_J \log\cosh(\beta J)\right) + \frac{z}{2N}T\ \mathbb{E}_J \log\cosh(\beta J) - \frac{T}{2N}\sum_{L=3}^{\infty}\frac{z^L}{L}\mathbb{E}_{\{J_i\}}\log\left[1+\prod_{i=1}^{L}\tanh(\beta J_i)\right], \quad (39)$$

thus recovering the simplified replica result of Eq. (26).

We can conclude that the replica calculation reproduces correctly all the topological structures involved in the $1/N$ corrections to the free-energy density. Incidentally, we note that self-intersecting loops occur only with probability N^{-2}, and they do not contribute to $1/N$ corrections.

Let us finish this section by giving a different interpretation to the present results. We have seen that under the assumption that finite-size corrections can be computed by the cavity method in a graph with finite densities of certain topological structures, we have been able to reproduce the replica result (and give to it a more physical intuition). However, we could assume that replica and cavity methods should provide the correct free energy for a very large, but finite, system and could then conclude that the free energy of a model only depends on the densities of certain topological structures. This alternative view can be useful if one aims at computing the free energy of a model defined on a finite-dimensional lattice by considering a lattice to be a random graph with strong topological correlations and making an expansion in these topological correlations (e.g., number and size of loops).

V. NUMERICAL ANALYSIS

In this section we check the validity of our analytical expressions for the free-energy corrections, Eq. (25), against numerical simulations. Since from Monte Carlo simulations one obtains the energy of the systems, in order to avoid an integration in temperature we decided to perform the simulations at zero temperature, where energy and free energy coincide. Moreover, since Eq. (25) holds for arbitrary disorder in the interaction and in the external field, we choose to keep the former deterministic and the latter randomly distributed. In this case, in fact, an exact polynomial algorithm is available to calculate the ground state. Therefore we apply Eq. (25) to compute the finite-size corrections to the energy density of the zero-temperature random-field Ising model (zt-RFIM) and compare them with numerical simulations. The model is defined by the following Hamiltonian:

$$\mathcal{H} = -J\sum_{i,j} C_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \quad (40)$$

where the random magnetic fields are Gaussian random variables of zero mean and variance $\overline{h_i^2} = 1$ and the ferromagnetic exchange coupling J take values in the interval $[0,\infty)$. The underlying graph topology is that of an Erdös-Rényi random graph. Due to the Fortuin-Kasteleyn-Ginibre (FKG)[23] inequality the model does not undergo replica-symmetry breaking[4] at any value of the ferromagnetic interaction strength J, so our formulas for the finite-size (free) energy density corrections remain valid also below the critical point, provided that a single pure state is selected. In the ferromagnetic phase the existence of two energy minima generates additional finite-size fluctuations, which are proportional to $N^{-1/2}$. These kinds of *interstate* fluctuations overcome the $1/N$ *intrastate* contribution, which becomes practically invisible in numerical experiments. In this work we compare analytical predictions and numerical results only in the paramagnetic phase $J < J_c$.

The uniqueness of the ground state of the model allows us to translate formula (25) for the free-energy density corrections into the corresponding expression for the ground-state energy density corrections. We write the ground-state energy density as the leading term plus the $O(\frac{1}{N})$ correction:

$$e^{\text{GS}}(N) = e_0^{\text{GS}} + \frac{1}{N}e^{(1)} + o\left(\frac{1}{N}\right), \quad (41)$$

where $e^{(1)}$ reads

$$e^{(1)} = -\left(z - \frac{z^2}{2}\right)\overline{|h^c|} - \frac{z}{2}e_1^a - \frac{z^2}{2}\left(e_2^a - 2e_1^a\right) + \frac{1}{2}\sum_{L=3}^{\infty}\frac{z^L}{L}\left[e_L^c - L\left(e_L^a - e_{L-1}^a\right)\right]. \quad (42)$$

The random variable h^c is the cavity field, distributed according to the zero-temperature solution of Eq. (16), while $e_L^{a/c}$ are the energies of open and closed chains in the graph. The computational time cost of computing the energy density of a chain of size L by enumeration is exponentially increasing in L; therefore only partial sums up to $L = 7$ in Eq. (42) have been considered in Fig. 1. To accurately compute the whole L series, especially near the critical point, some assumptions have to be made about the large-L behavior of its term. Some of the authors have been developing a formalism through which a spectral representation of the replicated transfer matrix[24–26] can be obtained. Using this result, the leading behavior

$$e_L^c - L\left(e_L^a - e_{L-1}^a\right) \sim AL\lambda^L \quad (43)$$

has been established for the zero-temperature RFIM, which allows us to analytically sum the remaining terms of the series (from $L = 8$ to infinity). Coefficient λ is given by the second eigenvalue of the replicated transfer matrix and gives the decay rate of ferromagnetic correlation functions. It can be computed to high precision with population dynamics techniques or as the first eigenvalue of an integral operator.

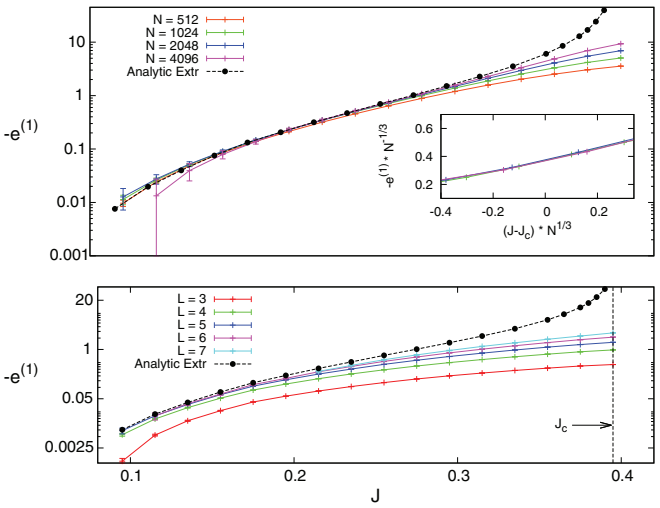

FIG. 1. (Color online) Finite-size corrections of the ground-state energy density in the $T = 0$ RFIM on Erdös-Rényi random graphs with mean connectivity $z = 4$. (top) Numerical data for different system sizes and the analytical formula given by Eq. (42). Close to the critical point (which is $J_c \approx 0.395$) the scaling of the energy corrections is given by the mean-field prediction (45), as confirmed by the data collapse shown in the inset. (bottom) Estimates of formula (42), truncating the sum over loops with a cutoff $L = 3,4,5,6,7$ and extrapolating the whole series as explained in the main text.

Coefficient A instead has been obtained from a fit of the first five points of the series. As an alternative approach assuming the validity of the $AL\lambda^L$ behavior (which fares much better than a simple exponential decay assumption), both A and λ could be inferred from a fit of the first terms of the sum. The finite-size corrections of the energy in the RFIM at zero temperature diverge as $e^{(1)} \propto \frac{1}{1-z\lambda}$, at odds with the double-pole divergence $e^{(1)} \propto \frac{1}{(1-z\lambda)^2}$ which can be found at finite temperature. This matter will be elucidated in a future work.[24]

At the critical point a scaling analysis of the correction $e^{(1)}$ can be performed. Calling $\tau = |J - J_c|$ the distance from the critical point, mean-field theory[27] predicts the following finite-size scaling for τ and $e^{(1)}$ in the critical region:

$$\tau = \frac{\tilde{\tau}}{N^{1/3}}, \tag{44}$$

$$e^{(1)} = \tilde{e}^{(1)} N^{1/3}. \tag{45}$$

The leading correction to the thermodynamic ground-state energy density is of order $O(N^{-2/3})$ in the whole critical region:

$$e^{\mathrm{GS}}(N) = e_0^{\mathrm{GS}} + \frac{1}{N^{2/3}}\tilde{e}^{(1)} + o\left(\frac{1}{N^{2/3}}\right) \quad \text{for} \quad J \to J_c\ . \tag{46}$$

Furthermore Eq. (41) is not valid in the ferromagnetic phase (for reasons mentioned in the beginning of this section), where the leading correction happens to be of order $O(N^{-1/2})$:

$$e^{\mathrm{GS}}(N) = e_0^{\mathrm{GS}} + \frac{1}{N^{1/2}}e'^{(1)} + o\left(\frac{1}{N^{1/2}}\right) \quad \text{for} \quad J > J_c\ .$$

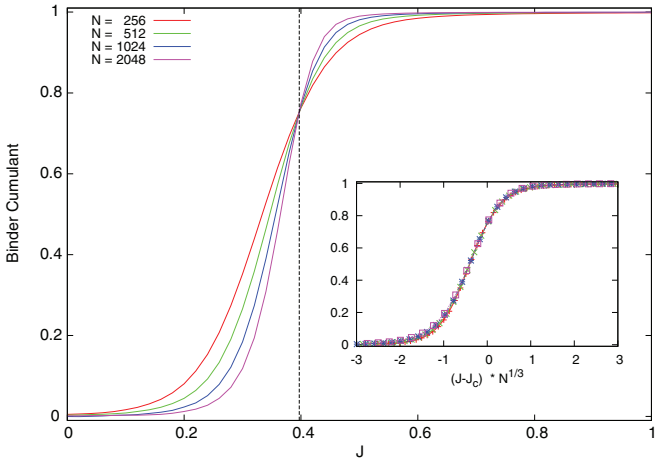

FIG. 2. (Color online) Binder cumulant for the $T = 0$ RFIM on Erdös-Rényi random graphs with mean connectivity $z = 4$ for different system sizes as a function of the exchange interaction J. A vertical dashed line is drawn in correspondence to the critical point $J_c \sim 0.395$. The inset shows the data collapse in the critical region using the scaling variable $(J - J_c)N^{1/3}$ for the reduced interaction.

(47)

The numerical experiment is performed on an Erdös-Rényi random graph with average connectivity $z = 4$. We compute the ground-state energy with the minimum-cut algorithm,[28,29] using the LEMON library.[30] To draw the profiles of the energy density corrections in Fig. 1 we took the average over 10^8 samples for each system size. Also in Fig. 1 we compare the numerical data with the analytical prediction given by Eq. (42) and check the finite-size scaling relation given by Eq. (45). In Fig. 2 we report the Binder cumulant:

$$\mathrm{Bi} = \frac{3}{2}\left[1 - \frac{\overline{m^4}}{3(\overline{m^2})^2}\right] \tag{48}$$

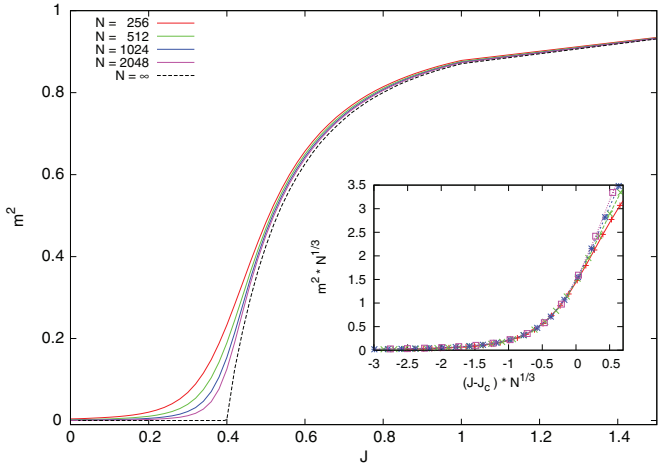

FIG. 3. (Color online) Average squared magnetization for the $T = 0$ RFIM on Erdös-Rényi random graphs with mean connectivity $z = 4$ for different system sizes as a function of the exchange interaction J. The inset shows the data collapse in the critical region using the scaling $N^{\frac{1}{3}}$ for both the reduced interaction and the squared magnetization.

for system sizes ranging from $N = 256$ to $N = 2048$. From the intersection of the curves we identify the critical point, obtaining $J_c \sim 0.395(1)$.

Figure 3 shows the behavior of the averaged squared magnetization $\overline{m^2}$. The finite-size scaling of $\overline{m^2}$ in the critical region is given by the following scaling relation:

$$\overline{m^2} = O(N^{-1/3}) = O(\tau) \quad \text{for} \quad \tau \to 0. \tag{49}$$

This scaling form is confirmed by the data collapse shown in the inset of Fig. 3.

VI. SUMMARY AND CONCLUSIONS

In this work we performed a thorough analysis of the $O(\frac{1}{N})$ correction to the free-energy density in disordered Ising models defined on Erdös-Rényi random graphs. We derived an analytical formula which can be easily used to quantify finite-size effects, avoiding the subtleties associated with the diagonalization of the Hessian. We also checked the correctness of our results through a numerical study of the RFIM at zero temperature and found that the finite-size corrections to the ground-state energy density are in perfect agreement with the analytical prediction.

More care has to be paid when studying finite-size corrections in the ferromagnetic ordered phase. The formulas derived in this work are intended to be correct only where a single pure state is concerned since they represent fluctuations inside a single pure state. Below the critical point, the continuous appearance of a couple of equivalent pure states generates additional *interstate* fluctuations, which cannot be described by formula (25). The nature of the low-temperature finite-size corrections is nonperturbative, so a different approach has to be taken in order to quantify them. Heuristic arguments and numerical simulations suggest that the first term of the free-energy expansion is $O(N^{-1/2})$ [at variance with the normal $O(\frac{1}{N})$ behavior], which dominates the *intrastate* contribution. Analogously, when exponentially many pure states are involved, as in the case of spin glasses in their glassy phase, we expect the leading finite-size correction to be much bigger than $O(\frac{1}{N})$ and expressions (24) and (25) to no longer hold. We also showed how replica results for the $1/N$ corrections to the free-energy density can be derived also in the cavity formalism, resorting to an auxiliary graph ensemble which in some sense lifts the $O(\frac{1}{N})$ *contributions* to the leading order. It would be interesting to see if this combinatorial derivation could be transposed to other graph ensembles. Moreover we expect our main result (25) to hold some degree of universality, depending only on a few topological properties, such as the mean residual degree z.

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APPENDIX A: COEFFICIENTS a_L

Coefficients a_L can be determined by computing recursively all functions S_L. A little bit of thought should convince us that S_L is given by the probability of the event $E_L \equiv$ {L consecutive links are present} by splitting it into at least two smaller events E_{L_1} and E_{L_2}, with $L_1 + L_2 = L$. Since $p_L = \text{Prob}[E_L]$, it should be clear that $q_L \equiv p_L - S_L$ is like a "connected" probability to obtain the L links from a unique structure. It is not hard to derive a recursive equation for functions S_L, valid for any $L \geqslant 2$:

$$S_L = q_1 p_{L-1} + q_2 p_{L-2} + \cdots + q_{L-1} p_1, \tag{A1}$$

from which we get, for any $L \geqslant 1$,

$$p_L(1 + q_0) = \sum_{k=0}^{L} q_k p_{L-k}, \tag{A2}$$

where $q_0 \equiv 0$ and $p_0 = 1$ thanks to the fact that $p_L = z^L[1 - L(L+1)/2N]$. The above equation can be easily solved by introducing the generating functions:

$$p(x) \equiv \sum_{k=0}^{\infty} p_k x^k, \quad q(x) \equiv \sum_{k=0}^{\infty} q_k x^k, \tag{A3}$$

which must satisfy

$$(1 + q_0)[p(x) - p_0] = q(x)p(x) - q_0 p_0 \implies q(x) = p_0 - \frac{1}{p(x)}. \tag{A4}$$

Keeping only terms up to order $1/N$, the result is

$$q(x) = zx - \frac{1}{N}\frac{zx}{1 - zx}, \tag{A5}$$

implying

$$q_1 = z\left(1 - \frac{1}{N}\right), \quad q_k = -\frac{z^k}{N} \quad \text{for} \quad k \geqslant 2. \tag{A6}$$

Rewriting Eq. (A2) as

$$q_L = \sum_{k=L}^{\infty}(k - L + 1)a_k + \sum_{k=L+1}^{\infty} c_k, \tag{A7}$$

we can obtain

$$q_L - q_{L+1} = \sum_{k=L}^{\infty} a_k + c_{L+1} \implies \sum_{k=L}^{\infty} a_k = q_L = -\frac{z^L}{N} \quad \forall\, L \geqslant 2 \tag{A8}$$

by noticing that $c_L = -q_L$ for $L \geqslant 3$. Moreover, for $L = 1$ we have

$$\sum_{k=1}^{\infty} a_k = q_1 - q_2 = z + \frac{z^2 - z}{N}. \tag{A9}$$

In conclusion coefficients a_L are given by the following expressions:

$$a_1 = z + \frac{1}{N}(2z^2 - z), \tag{A10}$$

$$a_L = \frac{1}{N}(z^{L+1} - z^L) \quad \text{for} \quad L \geqslant 2. \tag{A11}$$

APPENDIX B: COMBINATORICS OF $\mathrm{Tr}(T^L)$

Here we prove Eq. (23), relating $\mathrm{Tr}(T^L)$ in the small-n limit to the free energies of open and closed cavity chains. Let's rewrite our $2^n \times 2^n$ matrix as

$$T(\sigma,\sigma') = z\,\mathbb{E}\left\{\frac{1}{[2\cosh(\beta h)]^n}\left[e^{\beta J\sum_a \sigma_a\sigma'_a+\beta h\sum_a \sigma'_a} - \frac{1}{[2\cosh(\beta h')]^n}\left(\int d\tau\, e^{\beta J\sum_a \sigma_a\tau_a+\beta h\sum_a \tau_a}\right) e^{\beta h'\sum_a \sigma'_a}\right]\right\}. \tag{B1}$$

where expectation is taken over the coupling J and the cavity fields h,h', which are distributed according to the solution of Eq. (16). We immediately note that the factor $[2\cosh(\beta h)]^n$ reduces to $1 + n\log 2\cosh(\beta h) + o(n)$ in the small-n limit, allowing us to rewrite $T(\sigma,\sigma')$, with $o(n)$ accuracy, as

$$T(\sigma,\sigma') = z\,\mathbb{E}\left[e^{\beta J\sum_a \sigma_a\sigma'_a+\beta h\sum_a \sigma'_a} - \left(\int d\tau\, e^{\beta J\sum_a \sigma_a\tau_a+\beta h\sum_a \tau_a}\right) e^{\beta h'\sum_a \sigma'_a}\right] + nz\mathbb{E}_h \log 2\cosh(\beta h) + o(n). \tag{B2}$$

We recognize that the term $\mathbb{E}[e^{\beta J\sum_a \sigma_a\sigma'_a+\beta h\sum_a \sigma'_a}]$ is the replicated transfer matrix of a one-dimensional chain, and so when we take the trace of $T(\sigma,\sigma')$, we simply get

$$\mathrm{Tr}[T] = -n\beta z\left\{\phi_1^c - \left[\phi_1^a + \beta^{-1}\mathbb{E}_h \log 2\cosh(\beta h)\right]\right\} + o(n). \tag{B3}$$

When computing the trace of T^L, for $L \geqslant 2$, the term $nz\mathbb{E}_h \log 2\cosh(\beta h)$ in Eq. (B2) gives only contributions of order $o(n)$ and thus can be completely neglected in the following calculation. Let's denote $A = \mathbb{E}[e^{\beta J\sum_a \sigma_a\sigma'_a+\beta h\sum_a \sigma'_a}]$ and $B = \mathbb{E}[(\int d\tau\, e^{\beta J\sum_a \sigma_a\tau_a+\beta h\sum_a \tau_a})e^{\beta h\sum_a \sigma'_a}]$. The product T^L is formed by a linear combination of all the possible products of L matrices chosen between A and B; therefore we now consider the traces of such products. A simple inspection shows immediately $\mathrm{Tr}(A^L)$ is nothing more than the replicated partition function of a cavity loop, that is, a closed chain of length L embedded in a locally treelike random graph. Consider instead a term with one insertion of the matrix B, $\mathrm{Tr}A\cdots ABA\cdots$. Since B is factorized, its insertion prevents the closure of the chain, and we obtain the replicated partition function of an open cavity chain of length L. Generalizing the argument, we can see that the trace of a product containing k matrices B yields the product of k replicated partition functions of open chains, whose total lengths adds up to L. Since in the $n \downarrow 0$ limit products of partition functions become the sum of free energies, we can write

$$\frac{\partial}{\partial n}\mathrm{Tr}(T^L) = -\beta z^L\left[\phi_L^c + \sum_{l=1}^{L} b_l\,\phi_l^a\right] + o(1), \tag{B4}$$

where coefficients b_l have to be determined. It is easy to see that $b_L = -L$ and $b_{L-1} = L$, while a simple combinatoric argument gives the remaining coefficients. We can construct an open chain of length $l < L-1$ in the first $l+1$ positions of the product and then multiply for the L possible ways of obtaining the same trace. So we consider products of the form $BA^{l-1}B \times \{2^{L-l-1}$ different combinations of A and B$\}$. Taking into account the number of insertions of B in the last $L-l-1$ positions, we obtain

$$b_l = L \times \sum_{k=0}^{L-l-1}(-1)^k\binom{L-l-1}{k} = 0 \quad \text{for} \quad l < L-1\,, \tag{B5}$$

which immediately yields Eq. (23).

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